Problem Sheet 9

November 10, 2025

Question 1

Let $T > 0$, $u_0 : [0,1] \to \mathbb{R}$ and $u : [0,1] \times [0,T] \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) = 0, & 0 < x < 1,\ 0 < t \leqslant T; \\ u(0,t) = 0, & 0 < t \leqslant T; \\ u(1,t) = 0, & 0 < t \leqslant T; \\ u(x,0) = u_0(x), & 0 < x < 1. \end{cases} \tag{1}$$

We assume u_0 and u are smooth and we want to prove that if $u_0(x) \geqslant 0\ \forall x \in [0,1]$ then $u(x,t) \geqslant 0\ \forall (x,t) \in [0,1]\times[0,T]$. Let $u^-(x,t) := \max\{-u(x,t), 0\}$. Prove that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (u^-(x,t))^2 dx + \int_0^1 (\frac{\partial u^-}{\partial x}(x,t))^2 dx = 0.$$

Hint : multiply the heat equation by $u^-(x,t)$ and integrate between $x = 0$ and $x = 1$.

Question 2

Consider the time dependent advection-diffusion problem : Given $T, \epsilon > 0$, find $u : [0,1] \times [0,T] \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \epsilon\dfrac{\partial^2 u}{\partial x^2}(x,t) - \dfrac{\partial u}{\partial x}(x,t) = 1, & 0 < x < 1,\ t > 0; \\ u(0,t) = 0, & t > 0; \\ u(1,t) = 0, & t > 0; \\ u(x,0) = 0, & 0 < x < 1. \end{cases} \tag{2}$$

Let M, N be large integers, $h = \dfrac{1}{N+1}$, $\tau = \dfrac{T}{M}$, $x_i = ih$, $i = 0, 1, \ldots, N+1$, $t_n = n\tau$, $n = 0, 1, \ldots, M$.

(a) Proceed as in the lecture to write a difference scheme (write the PDE at time t_{n+1}, use a backward Finite Difference Formula for $\dfrac{\partial u}{\partial t}$ and a centered FDF for $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial^2 u}{\partial x^2}$).
Write the scheme as

$$(I + \tau A)\vec{u}^{n+1} = \tau\vec{1} + \vec{u}^n,\ n = 0, 1, \ldots, M-1,$$

where $\vec{u}^n$ has components u_i^n, $i = 1, \ldots, N$, which are approximations of $u(x_i, t_n)$ and where A has to be defined.

Assume $h \leqslant 2\epsilon$.

(b) Prove that $||\vec{u}^{n+1}||_\infty \leqslant ||\vec{u}^n||_\infty + \tau$, so that $||\vec{u}^M||_\infty \leqslant T$.

(c) Prove that $u_i^n \geqslant 0$, $i = 1, \ldots, N$, $n = 1, \ldots, M$.

(d) Let $\vec{U}^n \in \mathbb{R}^N$ with components $u(x_i, t_n)$. Prove that, under reasonable assumptions on u,

$$(I + \tau A)\vec{U}^{n+1} = \tau\vec{1} + \vec{U}^n + \tau\vec{r}^{n+1},$$

where $||\vec{r}^{n+1}||_\infty \leqslant C(h^2 + \tau)$, with C independent of h and τ.

(e) Prove that $||\vec{U}^M - \vec{u}^M||_\infty \leqslant CT(h^2 + \tau)$.

Question 3

Graded Exercise for Group 3

Implement the scheme studied in Question 2. The Matlab file heat.m implements the heat equations, modify the file as required.

Check convergence : fix an $\epsilon = 0.01$ and play with the space and time step to provide a three digits approximation of $u(x = 0.9, t = 0.5)$.

What happens when :

— $T = 100,\ \epsilon = 0.01,\ N = 99,\ M = 100$?

— $T = 100,\ \epsilon = 0.01,\ N = 19,\ M = 100$?

EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2025

Answer Key 9

November 10, 2025

Question 1

Following the hint,

$$\int_0^1 \frac{\partial u}{\partial t}(x,t)u^-(x,t)dx - \int_0^1 \frac{\partial^2 u}{\partial x^2}(x,t)u^-(x,t)dx = 0,$$

integrate by part the second term,

$$\int_0^1 \frac{\partial u}{\partial t}(x,t)u^-(x,t)dx + \int_0^1 \frac{\partial u}{\partial x}(x,t)\frac{\partial u^-}{\partial x}(x,t)dx = 0.$$

Since $\int_0^1 \frac{\partial u}{\partial t}(x,t)u^-(x,t)dx = -\int_0^1 \frac{\partial u^-}{\partial t}(x,t)u^-(x,t)dx$ and $\int_0^1$ $frac\partial u\partial x(x,t)\frac{\partial u^-}{\partial x}(x,t)dx = -\int_0^1 \frac{\partial u^-}{\partial x}(x,t)\frac{\partial u^-}{\partial x}(x,t)dx$ we have,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (u^-(x,t))^2dx + \int_0^1 (\frac{\partial u^-}{\partial x}(x,t))^2dx = 0.$$

Thus for any $T > 0$: $\int_0^1 (u^-(x,T))^2dx = \int_0^1 (u^-(x,0))^2dx$ and since $u_0(x) \geqslant 0$ then $\int_0^1 (u^-(x,0))^2dx = 0$ and $\int_0^1 (u^-(x,t))^2dx = 0$ $\forall t > 0$ and $u^-(x,t) = 0$.

Question 2

(a) The difference scheme reads

$$\begin{cases} \dfrac{u_i^{n+1}-u_i^n}{\tau} - \epsilon\dfrac{u_{i-1}^{n+1}-2u_i^{n+1}+u_{i+1}^{n+1}}{h^2} - \dfrac{u_{i+1}^{n+1}-u_{i-1}^{n+1}}{2h} = 1, & n=1,\ldots,M-1,\ i=1,\ldots,N; \\ u_0^n = 0, & n=1,\ldots,M-1; \\ u_{N+1}^n = 0, & n=1,\ldots,M-1; \\ u_i^0 = 0, & i=0,\ldots,N+1. \end{cases} \tag{1}$$

(b) Let k be such that $|u_k^{n+1}| \geqslant |u_i^{n+1}|$ for every $i = 1,\ldots,N$. Then,

$$\Big(1+\frac{2\tau\epsilon}{h^2}\Big)u_k^{n+1} = u_k^n + \Big(-\frac{\tau}{2h}+\frac{\tau\epsilon}{h^2}\Big)u_{k-1}^{n+1} + \Big(\frac{\tau}{2h}+\frac{\tau\epsilon}{h^2}\Big)u_{k+1}^{n+1} + \tau.$$

Since $h \leqslant 2\epsilon$, this implies

$$\Big(1+\frac{2\tau\epsilon}{h^2}\Big)|u_k^{n+1}| \leqslant |u_k^n| + \Big(-\frac{\tau}{2h}+\frac{\tau\epsilon}{h^2}\Big)|u_k^{n+1}| + \Big(\frac{\tau}{2h}+\frac{\tau\epsilon}{h^2}\Big)|u_k^{n+1}| + \tau$$

and therefore

$$||u^{n+1}||_\infty = |u_k^{n+1}| \leqslant |u_k^n| + \tau \leqslant ||u^n||_\infty + \tau.$$

(c) Let k such that $u_k^{n+1} \leqslant u_i^{n+1}$, for every $i = 1,\ldots,N$. It suffices to show that $u_k^{n+1} \geqslant 0$. Then

$$\begin{aligned}\Big(1+\frac{2\tau\epsilon}{h^2}\Big)u_k^{n+1} &= u_k^n + \Big(-\frac{\tau}{2h}+\frac{\tau\epsilon}{h^2}\Big)u_{k-1}^{n+1} + \Big(\frac{\tau}{2h}+\frac{\tau\epsilon}{h^2}\Big)u_{k+1}^{n+1} + \tau \\ &\geqslant u_k^n + \Big(-\frac{\tau}{2h}+\frac{\tau\epsilon}{h^2}\Big)u_k^{n+1} + \Big(\frac{\tau}{2h}+\frac{\tau\epsilon}{h^2}\Big)u_k^{n+1} + \tau\end{aligned}$$

and thus

$$u_k^{n+1} \geqslant u_k^n + \tau \geqslant 0.$$

(d) Assume that $u(x,t)$ is $\mathcal{C}^4$ in space and $\mathcal{C}^2$ in time. Then

$$\frac{U_i^{n+1}-u_i^n}{\tau} - \epsilon\frac{U_{i-1}^{n+1}-2U_i^{n+1}+U_{i+1}^{n+1}}{h^2} - \frac{U_{i+1}^{n+1}-U_{i-1}^{n+1}}{2h} = 1 + r_i^{n+1},$$

with

$$r_i^{n+1} \leqslant \frac{h^2}{12}\max_{(x,t)\in[0,1]\times[0,T]}|\frac{\partial^4 u}{\partial x^4}(x,t)| + \frac{\tau}{2}\max_{(x,t)\in[0,1]\times[0,T]}|\frac{\partial^2 u}{\partial t^2}(x,t)| + \frac{h^2}{6}\max_{(x,t)\in[0,1]\times[0,T]}|\frac{\partial^3 u}{\partial x^3}(x,t)|.$$

(e) Setting $\vec{e}^n = \vec{U}^n - \vec{u}^n$, we have

$$(I+\tau A)\vec{e}^{n+1} = \vec{e}^n + \tau\vec{r}^{n+1}$$

and thus

$$||\vec{e}^{n+1}||_\infty \leqslant ||\vec{e}^n||_\infty + \tau||\vec{r}^{n+1}||_\infty \leqslant ||\vec{e}^n||_\infty + \tau C(h^2+\tau),$$

which implies

$$||\vec{e}^M||_\infty \leqslant ||\vec{e}^0||_\infty + C\underbrace{\tau M}_{T}(h^2+\tau).$$

Question 3

Convergence,

M	N	$u(x=0.9, t=0.5)$
200	100	$9.379287e-02$
400	200	$9.392789e-02$
800	400	$9.399461e-02$
1600	800	$9.402778e-02$
3200	1600	$9.404431e-02$

As t grows, the solution converges to the solution of the diffusion convection problem seen in Problem Sheet 7 exercise 1, but reflected with respect to the y-axis since the advective term is with the minus sign. As for the diffusion convection

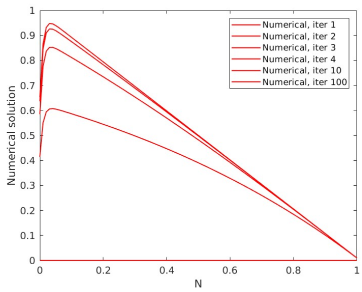

FIGURE 1 – Numerical solution for t=1,2,3,4,10,100.

problem, the numerical solution oscillates if h is too large (N too small).

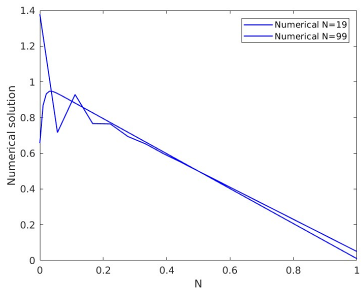

FIGURE 2 – Numerical solution for N=19,99.