Problem Sheet 8

November 3, 2025

Question 1

Graded exercise for group 2

Consider the 2D diffusion-convection problem :

$$-\epsilon\Delta u(x_1,x_2) + \frac{\partial u}{\partial x_1}(x_1,x_2) = 1 \ (x_1,x_2) \in]0,1[^2, \tag{1}$$

$$u(x_1,x_2) = 0 \ (x_1,x_2) \in \partial]0,1[^2. \tag{2}$$

implemented with order two centered finite differences schema.

- Fill the `diffconv2d.m` file to build the matrix of the linear system using only L (the number of points in each dimension), I (the identity $L \times L$ matrix) and E (the subdiagonal $L \times L$ matrix).
- Check the results when $\epsilon = 0.01$ and $L = 10, 20, 40, 80, 160$. Check that the number of iterations of the GMRES solver is $O(L)$.

Question 2

Let $\alpha > 0$, $T > 0$, $\beta \in C^0[0,1]$, $y \in C^1[0,1]$, such that for $0 \leqslant t \leqslant T$,

$$y'(t) + \alpha y(t) \leqslant \beta(t),$$

prove that

$$y(t) \leqslant y(0)e^{-\alpha t} + \int_0^t \beta(s)e^{-\alpha(t-s)}ds.$$

Hint : multiply by $e^{\alpha t}$.

Question 3

Let $T > 0$, $f : [0,1] \times [0,T] \to \mathbb{R}$, $u_0 : [0,1] \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. We consider the nonlinear heat problem : find $u : [0,1] \times [0,T] \to \mathbb{R}$ such that

$$\left\{ \begin{array}{rcll} \frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) & = & f(x,t) + g(u(x,t)) & \text{in } [0,1] \times [0,T], \\ u(x,0) & = & u_0(x) & \text{in } [0,1], \\ u(1,t) & = & u(0,t) = 0 & \text{in } [0,T]. \end{array} \right. \tag{3}$$

Assume $g(0) = 0$ and there exists L such that $\forall x, y \in \mathbb{R}$

$$|g(x) - g(y)| \leqslant L|x-y|.$$

- Show that
$$\frac{1}{2}\frac{d}{dt}\int_0^1 u(x,t)^2 dx + \int_0^1 \left(\frac{\partial u}{\partial x}(x,t)\right)^2 dx = \int_0^1 (f(x,t) + g(u(x,t)))u(x,t)dx.$$
- Show that
$$\frac{1}{2}\frac{d}{dt}\int_0^1 u(x,t)^2 dx + (1-2L)\int_0^1 u(x,t)^2 dx = \int_0^1 f(x,t)^2 dx.$$
- Conclude that :
$$\int_0^1 u(x,t)^2 dx \leqslant \int_0^1 u_0(x)^2 dx e^{-(1-2L)t} + \int_0^t \left(\int_0^1 f(x,t)^2 dx\right) e^{-(1-2L)(t-s)} ds.$$

Question 4

Definition : The Kronecker product of two $m \times m$ matrices B and C is the $m^2 \times m^2$ matrix

$$B \otimes C = \begin{bmatrix} b_{11}C & \cdots & b_{1m}C \\ \vdots & & \vdots \\ b_{m1}C & \cdots & b_{mm}C \end{bmatrix}.$$

A Kronecker sum of two $m \times m$ matrices B and C is the $m^2 \times m^2$ matrix

$$(I \otimes B) + (C \otimes I)$$

where I is the $m \times m$ identity matrix.

Show the following theorem :

Theorem : Let B and C be two $m \times m$ matrices whose the eigenvalues are $\lambda_1, ..., \lambda_m$ and $\mu_1, ..., \mu_m$ respectively. Then the eigenvalues of the Kronecker product are

$$\lambda_i \mu_j, \qquad i, j = 1, ..., m,$$

and the eigenvalues of the Kronecker sum are

$$\lambda_i + \mu_j, \qquad i, j = 1, ..., m.$$

Verify that $I_L \otimes A + A \otimes I_L$ has eigenvalues $4 - 2\left(cos\left(\frac{i\pi}{L+1}\right) + cos\left(\frac{j\pi}{L+1}\right)\right)$ and $1 \leqslant i, j \leqslant L$, where A is the tridiagonal matrix $(-1, 2, -1)$ of dimension $L \times L$.

Answer Key 8

November 3, 2025

Question 1

- Concerning the matlab file the A matrix is defined as :
  `A = (kron(T,I)+kron(I,T))*eps*(L+1)^2+kron(I,E'-E)*(L+1)/2.`
- The results for $\epsilon = 0.01$:

L	iteration_number
10	46
20	47
40	69
80	129
160	255

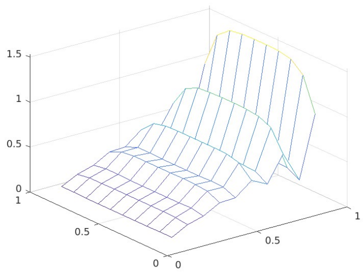

FIGURE 1 – Reasult for $\epsilon = 0.01$ and $L = 10$.

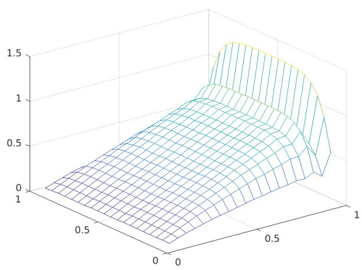

FIGURE 2 – Reasult for $\epsilon = 0.01$ and $L = 20$.

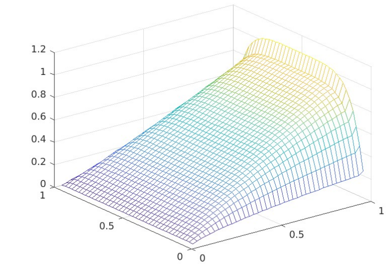

FIGURE 3 – Reasult for $\epsilon = 0.01$ and $L = 40$.

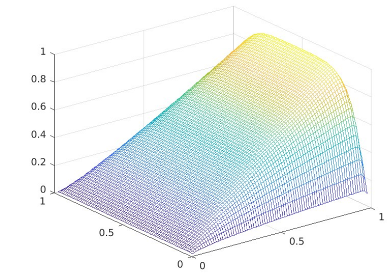

FIGURE 4 – Reasult for $\epsilon = 0.01$ and $L = 80$.

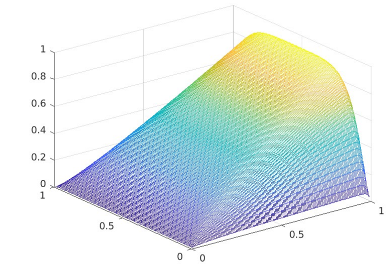

FIGURE 5 – Reasult for $\epsilon = 0.01$ and $L = 160$.

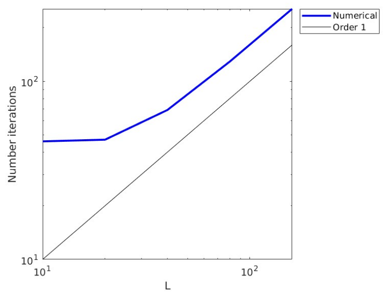

FIGURE 6 – Order of the number of iterations.

Question 2

Consider :

$$y'(t) + \alpha y(t) \leqslant \beta(t)$$

and multiply by $e^{\alpha t}$, then recognize the product rule : $\frac{d}{dt}(y(t)\alpha(t)) = y'(t)e^{\alpha t} + \alpha y(t)e^{\alpha t}$.
Finally by integrating in time you get to the conclusion :

$$y(t) \leqslant y(0)e^{-\alpha t} + \int_0^t \beta(s)e^{-\alpha(t-s)}ds.$$

Question 3

If we take $v = u$ as test function, we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u(x,t)^2dx + \int_0^1 (\frac{\partial u(x,t)}{\partial x})^2 dx = \int_0^1 (f(x,t) + g(u(x,t)))u(x,t)dx.$$

Let $C_P = 1$ denotes the constant in Poincaré inequality $\int_\Omega u^2 \leqslant \int_\Omega |\nabla u|^2$. Thanks to Cauchy-Schwarz and Young inequality and using the properties of g we obtain,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u(x,t)^2dx + \int_0^1 (u(x,t))^2dx \leqslant \frac{1}{2}\int_0^1 (f(x,t))^2dx + \frac{1}{2}\int_0^1 (u(x,t))^2dx + L\int_0^1 (u(x,t))^2dx.$$

that can be rewritten as :

$$\frac{d}{dt}\int_0^1 u(x,t)^2dx + (1-2L)\int_0^1 (u(x,t))^2dx \leqslant \int_0^1 (f(x,t))^2dx.$$

At this point write $y(t) = \int_0^1 u(x,t)^2dx$ and $\beta(t) = \int_0^1 (f(x,t))^2dx$, it is easy to see that we are in the same conditions of the Gronwall lemma, so it is enough to follow the steps done in Question 2 to get to the conclusion.

Question 4

Proof of the theorem :

If the vectors $\overrightarrow{u_i} = (u_{i1}, ..., u_{im})^T$, $i = 1, ..., m$, are the eigenvectors of B associated to the eigenvalues λ_i and the vectors $\overrightarrow{v_j} = (v_{j1}, ..., v_{jm})^T$, $j = 1, ..., m$, are the eigenvectors of C associated to the eigenvalues μ_j, then we have :

$$B \otimes C \begin{pmatrix} u_{i1}\overrightarrow{v_j} \\ \vdots \\ u_{im}\overrightarrow{v_j} \end{pmatrix} = \begin{pmatrix} b_{11}u_{i1}\mu_j\overrightarrow{v_j} + ... + b_{1m}u_{im}\mu_j\overrightarrow{v_j} \\ \vdots \\ b_{m1}u_{i1}\mu_j\overrightarrow{v_j} + ... + b_{mm}u_{im}\mu_j\overrightarrow{v_j} \end{pmatrix} = \begin{pmatrix} \lambda_i\mu_j u_{i1}\overrightarrow{v_j} \\ \vdots \\ \lambda_i\mu_j u_{im}\overrightarrow{v_j} \end{pmatrix}$$

and

$$((I \otimes B) + (C \otimes I)) \begin{pmatrix} v_{j1}\overrightarrow{u_i} \\ \vdots \\ v_{jm}\overrightarrow{u_i} \end{pmatrix} = \begin{pmatrix} v_{j1}\lambda_i\overrightarrow{u_i} \\ \vdots \\ v_{jm}\lambda_i\overrightarrow{u_i} \end{pmatrix} + \begin{pmatrix} c_{11}v_{j1}\overrightarrow{u_i} + ... + c_{1m}v_{jm}\overrightarrow{u_i} \\ \vdots \\ c_{m1}v_{j1}\overrightarrow{u_i} + ... + c_{mm}v_{jm}\overrightarrow{u_i} \end{pmatrix} = \begin{pmatrix} (\lambda_i + \mu_j)v_{j1}\overrightarrow{u_i} \\ \vdots \\ (\lambda_i + \mu_j)v_{jm}\overrightarrow{u_i} \end{pmatrix}$$

We easily verify that $A\overrightarrow{\phi_k} = \lambda_k\overrightarrow{\phi_k}$ for all $k \in \{1, ..., N\}$.

We have

$$\lambda_{max} = 2 - 2cos\frac{N\pi}{N+1} = 2 - 2cos\left(\pi - \frac{\pi}{N+1}\right) = 2 + 2cos\frac{\pi}{N+1}$$

and

$$\lambda_{min} = 2 - 2cos\frac{\pi}{N+1}.$$

We check that $I_L \otimes A + A \otimes I_L$. Thus the eigenvalues of the matrix are given by :

$$\lambda_i + \lambda_j = 4 - 2(cos\frac{i\pi}{L+1} + cos\frac{j\pi}{L+1}) \qquad i, j = 1, ..., L.$$