EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2025

Problem Sheet 7

November 3, 2025

Question 1

Let $\epsilon > 0$ (small), we are looking for $u : [0,1] \to \mathbb{R}$ such that

$$-\epsilon u''(x) + u'(x) = 1, \qquad 0 < x < 1, \tag{1}$$
$$u(0) = u(1) = 0. \tag{2}$$

— Check that $u(x) = x - \frac{1-e^{\frac{x}{\epsilon}}}{1-e^{\frac{1}{\epsilon}}}$ and plot the solution for $\epsilon = 0.01$.

— Let N be a positive integer, $h = \frac{1}{N+1}$, $x_i = ih$, $i = 0, 1, ..., N, N+1$. Consider the centered schema

$$-\epsilon \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \frac{u_{i+1} - u_{i-1}}{2h} = 1, \; i = 1, \dots, N, \tag{3}$$
$$u_0 = u_{N+1} = 0. \tag{4}$$

Write the schema as a linear system $A\vec{u} = \vec{1}$ (define A).

— Let $\vec{U}$ be the vector of exact solution $u(x_i)$, $i = 1, ..., N$. Prove that $A\vec{U} = \vec{1} + \vec{r}$ where $|r_i| \leqslant Ch^2$, $i = 1, ..., N$, with C independent of h.

— Prove that if $h \leqslant 2\epsilon$ then $A\vec{w} \geqslant \vec{1}$ where $w_i = x_i$, $i = 1, ..., N$.

— Prove that if $h \leqslant 2\epsilon$ then $A\vec{z} \geqslant \vec{0}$ implies $\vec{z} \geqslant \vec{0}$ (use Question 2 of Sheet 6).

— Let $\vec{g} \in R^N$ and $\vec{v}$ such that $A\vec{v} = \vec{g}$. Then prove that $\|\vec{v}\|_\infty \leqslant \|\vec{g}\|_\infty$.
Prove that there exists $\tilde{C} > 0$ such that $\forall 0 < h < 2\epsilon$

$$\|\vec{U} - \vec{u}\|_\infty \leqslant Ch^2. \tag{5}$$

— Implement the problem.

(a) For $\epsilon = 0.01$ and $N = 9, 19, 39, 79, 159, 319, \ldots$ check the convergence of the method.

(b) Take $N = 39$. What happens for $\epsilon = 0.1, 0.01, 0.001$?

Question 2

Graded exercise for group 1

Let $\epsilon > 0$ (small), we are looking for $u \in C^2([0,1])$ such that

$$-\epsilon u''(x) + u'(x) = 1, \qquad 0 < x < 1, \tag{6}$$
$$u(0) = u(1) = 0. \tag{7}$$

Prove that :

— $\forall v \in C^1([0,1])$, v(0) = v(1) = 0 holds $\int_0^1 (v(x))^2 \leqslant \int_0^1 (v'(x))^2$.

— Prove $\left(\int_0^1 (u'(x))^2\right)^{1/2} \leqslant \frac{1}{\epsilon}$.

— Write the finite difference schema as $(\epsilon A + B)\vec{u} = \vec{1}$ where : A is symmetric positive definite and $B = -B^T$. Prove that,

$$||\vec{u}||_2 \leqslant \frac{\sqrt{N}}{\epsilon \lambda_1},$$

where λ_1 is the smallest eigenvalue of A.

— Let $\vec{U} \in \mathbb{R}$ be the vector of components $u(x_i)$. From Question 1, we know that $(\epsilon A + B)\vec{U} = \vec{r}$, $||\vec{r}||_\infty \leqslant Ch^2$, with C depending only on u. Prove that,

$$||\vec{U} - \vec{u}||_2 \leqslant \frac{Ch^{3/2}}{\epsilon \lambda_1}$$

Question 3

The file `gradient.m` implements the gradient method with optimal step for the $L \times L$ matrix

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & -1 & 2 & -1 \end{pmatrix}.$$

Check that the number of iterations is $\mathcal{O}(L^2)$.

Question 4

The file `conjgrad.m` implements the Laplace problem in dimensions $d = 1, 2$ or 3 with finite differences, the linear system system being solved with the conjugate gradient method. L denotes the number of points in each direction.

Check the cpu time and memory requirement seen during the lecture and compare those with the Cholesky direct method. You can fill the following table :

	memory		cpu time		iteration_number
L	CG	Cholesky	CG	Cholesky	CG
10					
20					
40					
80					

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Answer Key 7

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Question 1

— A is given by :

$$A = \begin{pmatrix} \frac{2\epsilon}{h^2} & -\frac{\epsilon}{h^2} + \frac{1}{2h} & & & 0 \\ -\frac{\epsilon}{h^2} - \frac{1}{2h} & \frac{2\epsilon}{h^2} & & & \\ & \ddots & \ddots & \ddots & \\ 0 & & -\frac{\epsilon}{h^2} - \frac{1}{2h} & \frac{2\epsilon}{h^2} & -\frac{\epsilon}{h^2} + \frac{1}{2h} \\ & & & -\frac{\epsilon}{h^2} - \frac{1}{2h} & \frac{2\epsilon}{h^2} \end{pmatrix}.$$

— From 3 we can analyze separatly the two terms : $-\epsilon \frac{u_{i+1}-2u_i+u_{i-1}}{h^2}$ and $\frac{u_{i+1}-u_{i-1}}{2h}$. Concerning the first one holds the results seen during the lecture. Concerning the second one we proceed by first developing the Taylor expansions of u_{i+1} and u_{i-1} :

$$u(x_{i+1}) = u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + \frac{h^3}{6}u'''(\alpha_i) \ , x_i < \alpha_i < x_{i+i}, \tag{1}$$

$$u(x_{i-1}) = u(x_i) - hu'(x_i) + \frac{h^2}{2}u''(x_i) - \frac{h^3}{6}u'''(\beta_i) \ , x_{i-1} < \beta_i < x_i. \tag{2}$$

Then you take the difference :

$$\frac{u(x_{i+1}) - u(x_{i-1})}{2h} = u'(x_i) + \frac{h^3}{12h}(u'''(\alpha_i) + u'''(\beta_i)). \tag{3}$$

$$\tag{4}$$

Putting togheter the two terms for the residual you get that :

$$|r_i| \leqslant h^2 \left(\frac{\epsilon}{12} \max_{0 \leqslant x \leqslant 1} |u^{(4)}(x)| + \frac{1}{6} \max_{0 \leqslant x \leqslant 1} |u'''(x)|\right). \tag{5}$$

— We have :

$$-\epsilon \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + \frac{w_{i+1} - w_{i-1}}{2h} = 1, \ i = 1, \dots, N-1, \tag{6}$$

whereas on the last $w(N+1) = 1$:

$$-\epsilon \frac{w_{N-1} - 2w_N}{h^2} - \frac{w_{N-1}}{2h} = 1 + \frac{\epsilon}{h^2} - \frac{1}{2h} \geqslant 1 \tag{7}$$

if $\frac{\epsilon}{h^2} - \frac{1}{2h} \geqslant 0$.

— Check that you can use the Lemma of Question 2 Sheet 6 :

$$A = \begin{pmatrix} \alpha_1 & -\gamma_1 & & & 0 \\ -\beta_1 & \alpha_2 & -\gamma_2 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & -\beta_{N-2} & \alpha_{N-1} & -\gamma_{N-1} \\ & & & -\beta_{N-1} & \alpha_N \end{pmatrix}.$$

Since $\alpha_i = \frac{2\epsilon}{h^2} > 0$, $\beta_i = \frac{\epsilon}{h^2} + \frac{1}{2h} > 0$ and $\gamma_i = \frac{\epsilon}{h^2} - \frac{1}{2h} \geqslant 0$. Then $A\vec{z} \geqslant \vec{0}$ implies $\vec{z} \geqslant \vec{0}$.

— Since the maximum of $w_i = x_i$ is equal to 1. Then $\|\vec{g}\|_\infty = \max_{1\leqslant i\leqslant N} \vec{g} = 1$.
We have first to see if the Lemma saw during the lecture hold :

$$A\vec{w}\|\vec{g}\|_\infty = \vec{1}\|\vec{g}\|_\infty \tag{8}$$

also you have $A\vec{v} = \vec{g}$. Then :

$$A(\vec{w}\|\vec{g}\|_\infty - \vec{v}) = \vec{1}\|\vec{g}\|_\infty - \vec{g} \geqslant \vec{0}, A(\vec{w}\|\vec{g}\|_\infty + \vec{v}) = \vec{1}\|\vec{g}\|_\infty + \vec{g} \geqslant \vec{0}. \tag{9}$$

Thus : $-\vec{w}\|\vec{g}\|_\infty \leqslant \vec{v} \leqslant \vec{w}\|\vec{g}\|_\infty$.
That is : $|\vec{v}| \leqslant |\vec{w}|\|\vec{g}\|_\infty \leqslant \|\vec{g}\|_\infty$
Then taking the quantity $A(\vec{U} - \vec{u}) = \vec{r}$, by applying the Lemma you get :

$$\|\vec{U} - \vec{u}\|_\infty \leqslant \|\vec{r}\|_\infty \leqslant Ch^2, \tag{10}$$

where $C = \frac{\epsilon}{12} \max_{0\leqslant x\leqslant 1} |u^{(4)}(x)| + \frac{1}{6} \max_{0\leqslant x\leqslant 1} |u'''(x)|$.

— **(a)** We obtain the following results : .

N	9	19	39	79	159	319
$\left\|\vec{U} - \vec{u}\right\|_\infty$	$6.96 \cdot 10^{-1}$	$4.35 \cdot 10^{-1}$	$1.93 \cdot 10^{-1}$	$5.57 \cdot 10^{-2}$	$1.21 \cdot 10^{-2}$	$3.02 \cdot 10^{-3}$

As expected, $\left\|\vec{U} - \vec{u}\right\|_\infty = \mathcal{O}(h^2)$ when h is sufficiently small.

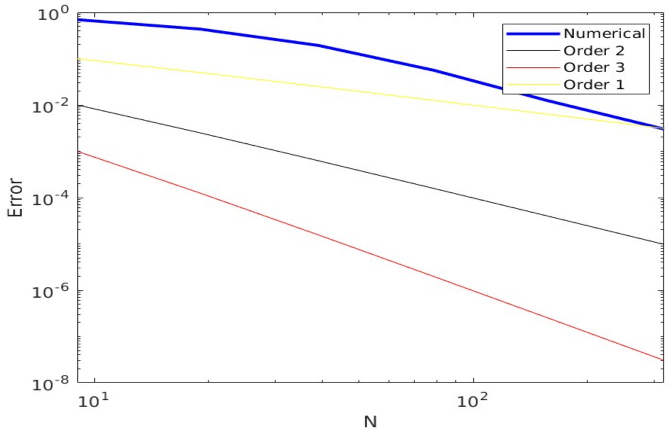

FIGURE 1 – Order of convergence.

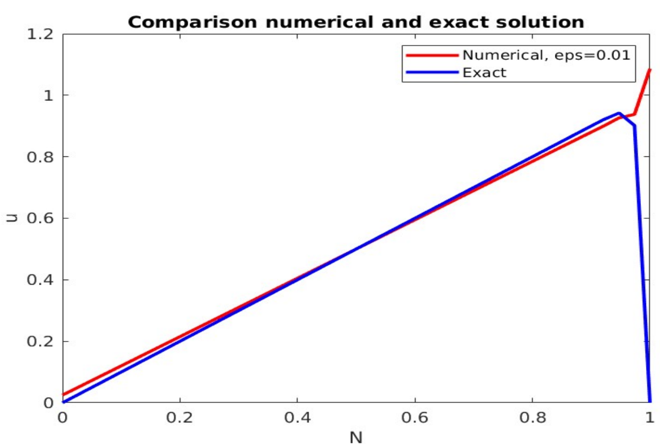

FIGURE 2 – Comparison of the real and numerical solution $\epsilon = 0.01$.

(b) For $N = 39$, the solution oscillates when $\epsilon = 0.001$.

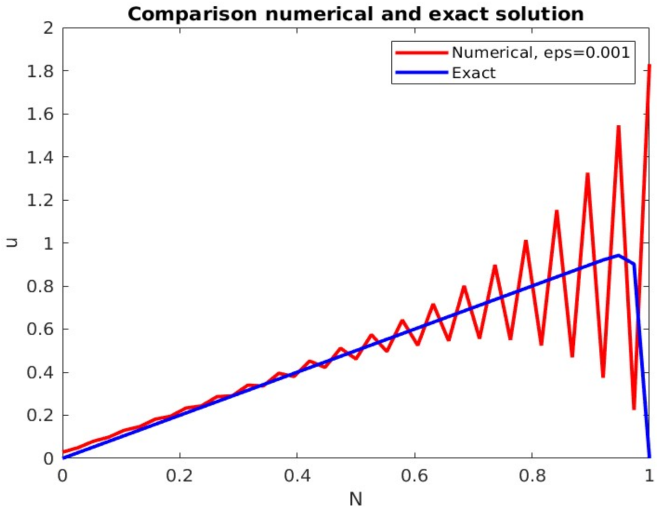

FIGURE 3 – Comparison of the real and numerical solution $\epsilon = 0.001$.

Question 2

— We assume $v(0) = 0$ The proof in the case $v(1) = 0$ is similar. Note that $\forall x \in [0, 1]$,

$$v(x) = v(0) + \int_0^x v'(s)ds = \int_0^x v'(s)ds.$$

Thus

$$(v(x))^2 = \left(\int_0^x v'(s)ds\right)^2 \leqslant \int_0^x (v'(s))^2 ds \int_0^x 1^2 ds \leqslant \int_0^1 (v'(s))^2 ds.$$

— Multiply the equation by u and integrate by part the first term,

$$\epsilon \int_0^1 (u')^2 dx + \underbrace{\int_0^1 u'udx}_{=0\text{, by applying the divergence theorem}} = \int_0^1 udx$$

by appling the Poincare inequality from point 1.,

$$\epsilon \int_0^1 (u')^2 dx \leqslant \left(\int_0^1 u^2 dx\right)^{1/2} \leqslant \left(\int_0^1 (u')^2 dx\right)^{1/2}$$

reording the terms you arrive to the conclusion.

— A is given by :

$$A = \begin{pmatrix} \frac{2\epsilon}{h^2} & -\frac{\epsilon}{h^2} & & & 0 \\ -\frac{\epsilon}{h^2} & \frac{2\epsilon}{h^2} & & & \\ & \ddots & \ddots & \ddots & \\ 0 & & -\frac{\epsilon}{h^2} & \frac{2\epsilon}{h^2} & -\frac{\epsilon}{h^2} \\ & & & -\frac{\epsilon}{h^2} & \frac{2\epsilon}{h^2} \end{pmatrix}.$$

and B is :

$$A = \begin{pmatrix} 0 & \frac{1}{2h} & & & 0 \\ -\frac{1}{2h} & 0 & & & \\ & \ddots & \ddots & \ddots & \\ 0 & & -\frac{1}{2h} & 0 & \frac{1}{2h} \\ & & & -\frac{1}{2h} & 0 \end{pmatrix}.$$

So you get the schema $(\epsilon A + B)\vec{u} = \vec{1}$. Then multiply the latter by $\vec{u}$: $\epsilon \vec{u}^T A \vec{u} + \vec{u}^T B \vec{u} = \vec{1}^T \vec{u}$. Then, because of the property that $B = -B^T$, the second term on the left-hand-side is zero. Finally we get,

$$\epsilon \lambda_1 ||\vec{u}||_2^2 \leqslant \epsilon \vec{u}^T A \vec{u} \leqslant ||\vec{1}||_2 ||\vec{u}||_2.$$

—

$$\epsilon\lambda_1||\vec{U}-\vec{u}||_2^2 \leqslant \epsilon(\vec{U}-\vec{u})^T A(\vec{U}-\vec{u}) + (\vec{U}-\vec{u})^T B(\vec{U}-\vec{u})$$
$$= \vec{r}^T(\vec{U}-\vec{u}) \leqslant ||\vec{r}||_2||\vec{U}-\vec{u}||_2.$$

finally,

$$\epsilon\lambda_1||\vec{U}-\vec{u}||_2 \leqslant ||\vec{r}||_2 \leqslant Ch^{3/2}.$$

Question 3

Running the code `gradient.m` for different values of L yields the table above, which corresponds to a $\mathcal{O}(L^2)$ growth.

L	10	20	40	80	160
iteration number	339	1349	5357	21505	86993

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Question 4

We obtain the following tables :

d = 1

	memory		cpu time		iteration_number
L	CG	Cholesky	CG	Cholesky	CG
10	536	392	0.000000e+00	5.000000e-02	5
20	1096	792	0.000000e+00	3.000000e-02	10
40	2216	1592	1.000000e-02	7.000000e-02	20
80	4456	3192	1.000000e-02	3.000000e-02	40
160	8936	6392	1.000000e-02	2.000000e-02	80

d = 2

	memory		cpu time		iteration_number
L	CG	Cholesky	CG	Cholesky	CG
10	9768	16952	1.000000e-02	2.000000e-02	14
20	40328	131512	1.000000e-02	6.000000e-02	32
40	163848	1037432	0.000000e+00	2.000000e-02	63
80	660488	8244472	7.000000e-02	2.000000e-02	127
160	2652168	65743352	1.890000e+00	1.400000e+00	255

d = 3

	memory		cpu time		iteration_number
L	CG	Cholesky	CG	Cholesky	CG
10	126408	1478552	1.000000e-02	8.000000e-02	20
20	1049608	48953912	5.000000e-02	1.040000e+00	41
40	8550408	1599975032	1.630000e+00	3.058000e+01	80
80	69017608	51793818872	7.880000e+00	1.105130e+03	162
160	554598408	-	7.459000e+01	-	327

The results we obtain indeed shows that the CPU times grows with relative order $\mathcal{O}\left(L^{d+1}\right) = \mathcal{O}\left(N^{\frac{d+1}{d}}\right)$. The CPU times are too small for $d = 1, 2$ to check the complexity. When $d = 3$ the cputime from $L = 80$ to $L = 160$ is multiply by $2^4 = 16$, which corresponds to $\mathcal{O}(L^4)$.

From the lecture we know that $K(A) = \dfrac{\lambda_N}{\lambda_1} = \mathcal{O}(L^2)$, thus the number of iter of CG is $\mathcal{O}(L)$ and thus the number of operations should be $\mathcal{O}(L^{d+1}) = \mathcal{O}(N^{\frac{d+1}{d}})$.