



### Question 3

Consider the problem, given  $f : [0, 1] \rightarrow \mathbb{R}$ , find  $u : [0, 1] \rightarrow \mathbb{R}$

$$\begin{cases} -u''(x) = f(x) & 0 < x < 1, \\ u(0) = 0 & u(1) = 0. \end{cases} \quad (3)$$

Let  $N$  be a positive integer,  $h = \frac{1}{N+1}$ ,  $x_i = ih$ ,  $i = 0, 1, \dots, N+1$ , and the schema (Numerov 1924)

$$\begin{cases} \frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = \frac{f(x_{i-1}) + 10f(x_i) + f(x_{i+1}))}{12} & i = 1, \dots, N, \\ u_0 = 0 & u_{N+1} = 0. \end{cases} \quad (4)$$

Prove that  $\forall f \in C^4[0, 1]$ ,  $\exists C > 0$ ,  $\forall 0 < h \leq 1$ ,

$$\max_{1 \leq i \leq N} |u(x_i) - u_i| \leq Ch^4. \quad (5)$$

# Answer Key 6

October 13, 2025

## Question 1

We have

$$u(x_N) = u(x_{N+1}) - hu'(x_{N+1}) + \frac{h^2}{2}u''(x_{N+1}) - \frac{h^3}{6}u'''(\alpha_{N+1}),$$

for some  $x_N < \alpha_{N+1} < x_{N+1}$ , and thus

$$\frac{u(x_{N+1}) - u(x_N)}{h} = \frac{h}{2}f(x_{N+1}) + \frac{h^2}{6}u'''(\alpha_{N+1}).$$

Therefore,  $A\vec{U} = \vec{f} + \vec{r}$ , with

$$\vec{r} = \begin{pmatrix} -\frac{h^2}{24}(u^{(4)}(\alpha_1) + u^{(4)}(\beta_1)) \\ \vdots \\ -\frac{h^2}{24}(u^{(4)}(\alpha_N) + u^{(4)}(\beta_N)) \\ \frac{h^2}{6}u'''(\alpha_{N+1}) \end{pmatrix},$$

where  $x_i < \alpha_i < x_{i+1}$ , and  $x_{i-1} < \beta_i < x_i$ ,  $i = 1 \dots, N$ .

Hence,  $\|\vec{r}\|_\infty \leq Ch^2$ , with  $C = \max\left(\max_{0 \leq x \leq 1} \frac{1}{12}|u^{(4)}(x)|, \max_{0 \leq x \leq 1} \frac{1}{6}|u'''(x)|\right)$ .

Then, note that A satisfies the assumption of Question 2, we thus conclude that  $A\vec{v} \geq 0 \Rightarrow \vec{v} \geq 0$ .

Using  $A\vec{w} \geq \vec{1}$  and proceeding next exactly as in the lecture, we get that if  $A\vec{v} = \vec{g}$ , then

$$\|\vec{v}\|_\infty \leq \|\vec{w}\|_\infty \|\vec{g}\|_\infty \leq \frac{3}{2} \|\vec{g}\|_\infty.$$

Since  $A(\vec{U} - \vec{u}) = \vec{r}$ , we conclude that

$$\|\vec{U} - \vec{u}\|_\infty \leq Ch^2.$$

## Question 2

Let  $z_k$  s.t.  $z_k \leq z_i$ ,  $i = 1, \dots, N$ . Then it suffices to show that  $z_k \geq 0$ .

- If  $k = 1$ ,  $(Az)_k \geq 0$  writes  $\alpha_1 z_1 \geq \gamma_1 z_1$  that is  $(\alpha_1 - \gamma_1)z_1 \geq 0$ .  
Since  $\alpha_1 > \gamma_1$  this implies  $z_1 \geq 0$ .
- If  $k = N$ ,  $(Az)_k \geq 0$  writes  $\alpha_N z_N \geq \beta_{N-1} z_{N-1}$ .  
If  $\alpha_N = \beta_{N-1}$ ,  $z_N \geq z_{N-1}$  thus  $z_N = z_{N-1}$ .  
If  $\alpha_N > \beta_{N-1}$ ,  $\alpha_N z_N \geq \beta_{N-1} z_N$  that is  $(\alpha_N - \beta_{N-1})z_N \geq 0$  so  $z_N \geq 0$ .
- If  $k \in 2, \dots, N-1$ ,  $(Az)_k \geq 0$  writes  $\alpha_k z_k \geq \beta_{k-1} z_{k-1} + \gamma_k z_{k+1} \geq \beta_{k-1} z_{k-1} + \gamma_k z_k$ .  
so  $(\alpha_k - \gamma_k)z_k \geq \beta_{k-1} z_{k-1}$ .  
We know  $\alpha_k - \gamma_k \geq \beta_{k-1}$ .  
If  $\alpha_k - \gamma_k = \beta_{k-1}$ ,  $z_k \geq z_{k-1}$  thus  $z_k = z_{k-1}$ .  
if  $\alpha_k - \gamma_k > \beta_{k-1}$ ,  $(\alpha_k - \gamma_k)z_k \geq \beta_{k-1} z_k$  that is  $(\alpha_k - \gamma_k - \beta_{k-1})z_k \geq 0$  so  $z_k \geq 0$ .

Repeat the process for  $k-1$  instead of  $k$  and so on until we reach  $k=1$ .

In all cases  $z_k \geq 0$ .

### Question 3

For  $i = 1, \dots, N$ ,

$$u(x_{i+1}) = u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + \dots + \frac{h^6}{6!}u^{(6)}(\alpha_i) \quad x_i < \alpha_i < x_{i+1}, \quad (1)$$

$$u(x_{i-1}) = u(x_i) - hu'(x_i) + \frac{h^2}{2}u''(x_i) + \dots + \frac{h^6}{6!}u^{(6)}(\beta_i) \quad x_{i-1} < \beta_i < x_i, \quad (2)$$

then,

$$f(x_i) = -u''(x_i) = \frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} + \underbrace{2\frac{h^2}{4!}u^{(4)}(x_i)}_{=f''(x_i)} + \frac{h^4}{6!}(u^{(6)}(\alpha_i) + u^{(6)}(\beta_i)), \quad (3)$$

by rearranging the terms,

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f(x_i) + 2\frac{h^2}{12}f''(x_i) + \frac{h^4}{6!}(f^{(4)}(\alpha_i) + f^{(4)}(\beta_i)). \quad (4)$$

Notice that,

$$f''(x_i) = \frac{f(x_{i-1}) - 2f(x_i) + f(x_{i+1}))}{h^2} - \frac{h^4}{4!}(f^{(4)}(\gamma_i) + f^{(4)}(\delta_i)) \quad x_i < \gamma_i < x_{i+1} \quad x_{i-1} < \delta_i < x_i. \quad (5)$$

Thus,

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = \frac{f(x_{i-1}) + 10f(x_i) + f(x_{i+1}))}{12} + r_i, \quad (6)$$

and

$$|r_i| \leq \frac{h^4}{6} \max_{0 \leq x \leq 1} |f^{(4)}(x)|.$$

Finally, setting  $\vec{U} = (u(x_1), \dots, u(x_N))'$  you get the system of equations  $A(\vec{U} - \vec{u}) = \vec{r}$  and

$$\|\vec{U} - \vec{u}\|_\infty \leq \frac{1}{8} \|\vec{r}\|_\infty \leq \underbrace{\frac{1}{48} |f^{(4)}(x)|}_{C} h^4. \quad (7)$$