

Problem Sheet 5

October 6, 2025

Question 1

Consider a particle of mass m attached to a spring of stiffness k . The position of the particle is denoted $X(t)$. Newton's equation, together with initial conditions, writes :

$$\begin{cases} mX''(t) + kX(t) = 0, & 0 \leq t \leq T, \\ X(0) = X_0, \\ X'(0) = V_0. \end{cases} \quad (1)$$

1. Check that

$$\frac{d}{dt} \left(\frac{1}{2}m(X'(t))^2 + \frac{1}{2}k(X(t))^2 \right) = 0.$$

2. Let N be the number of time steps, $h = \frac{T}{N}$ and $t_n = nh$ where $n = 0, 1, \dots, N$. Let X_n be the approximation of $X(t_n)$ obtained using the Newmark's scheme :

$$m \frac{X_{n+1} - 2X_n + X_{n-1}}{h^2} + k \frac{X_{n+1} + 2X_n + X_{n-1}}{4} = 0, \quad n = 1, \dots, N-1. \quad (2)$$

How can X_1 be computed ?

3. Prove that

$$\frac{1}{2}m \left(\frac{X_{n+1} - X_n}{h} \right)^2 + \frac{1}{2}k \left(\frac{X_{n+1} + X_n}{2} \right)^2 = \frac{1}{2}m \left(\frac{X_n - X_{n-1}}{h} \right)^2 + \frac{1}{2}k \left(\frac{X_n + X_{n-1}}{2} \right)^2.$$

Question 2

Let $f \in C^0([0, 1])$, $a \in C^0([0, 1])$, $a(x) > 0$ and $u \in C^2([0, 1])$ be such that

$$-\frac{d}{dx} \left(a(x) \frac{du}{dx}(x) \right) = f(x), \quad 0 < x < 1, \quad (3)$$

$$u(0) = u(1) = 0. \quad (4)$$

Let $N > 0$. $h = \frac{1}{N+1}$ and $x_i = ih$, $i = 0, 1, \dots, N+1$. Let u_i be the approximation value of $u(x_i)$, $i = 0, 1, \dots, N$. Consider the finite difference scheme

$$-\frac{1}{h} \left(a(x_{i+\frac{1}{2}}) \frac{u_{i+1} - u_i}{h} - a(x_{i-\frac{1}{2}}) \frac{u_i - u_{i-1}}{h} \right) = f(x_i), \quad i = 0, 1, \dots, N,$$

with $u_0 = 0 = u_{N+1}$.

— Check that A is symmetric positive definite.

— The matlab file diffin.m implements the method when $a(x) = 1$. Implement the scheme when $a(x) = 1 + x$ and $u(x) = x^{\frac{1-x}{2}}$. Check numerically that : $\max_{1 \leq i \leq N} |u(x_i) - u_i| = O(h^2)$

Question 3

Graded Exercise for Group 2

(a) Let $f \in C^0[0, 1]$ such that

$$-u''(x) + u'(x) + u(x) = f(x) \quad 0 < x < 1, \quad (5)$$

$$u(0) = 0 \quad u'(1) = 0 \quad (6)$$

and prove that,

$$\int_0^1 ((u'(x))^2 + (u(x))^2) dx + \frac{1}{2}(u(1))^2 = \int_0^1 f(x)u(x) dx. \quad (7)$$

(b) Let $\Omega \subset \mathbb{R}^2$ open, bounded, \vec{n} unit external normal, $\vec{a} \in \mathbb{R}^2$, $\vec{f} \in C^0(\bar{\Omega})$, $\Gamma_- = \{\vec{x} \in \partial\Omega; \vec{u} \cdot \vec{n} < 0\}$, assume there exists $u \in C^2(\bar{\Omega})$ such that

$$-\Delta \vec{u}(\vec{x}) + \vec{a} \cdot \nabla \vec{u}(\vec{x}) + \vec{u}(\vec{x}) = \vec{f}(\vec{x}) \quad \forall \vec{x} \in \Omega, \quad (8)$$

$$\vec{u}(\vec{x}) = 0 \quad \forall \vec{x} \in \Gamma_- \quad (9)$$

$$\frac{\partial \vec{u}(\vec{x})}{\partial \vec{n}} = 0 \quad \forall \vec{x} \in \partial\Omega/\Gamma_- \quad (10)$$

and prove that,

$$\int_{\Omega} (|\nabla \vec{u}(x)|^2 + (\vec{u}(x))^2) dx + \int_{\Omega/\Gamma_-} \vec{a} \cdot \vec{n} \frac{\vec{u}(x)^2}{2} ds = \int_{\Omega} \vec{f}(\vec{x})\vec{u}(\vec{x}) dx. \quad (11)$$

(c) Let $T > 0$, $\Omega \subset \mathbb{R}^2$ open, bounded, \vec{n} unit external normal, $\vec{a} \in \mathbb{R}^2$, $\vec{u}_0 \in C^0(\bar{\Omega})$, $\vec{f} \in C^0(\bar{\Omega})$, $\Gamma_- = \{\vec{x} \in \partial\Omega; \vec{u} \cdot \vec{n} < 0\}$, assume there exists $\vec{u} \in C^{2,1}(\bar{\Omega} \times [0, T])$ (C^2 in space and C^1 in time) such that

$$\frac{\partial \vec{u}}{\partial t}(\vec{x}, t) - \Delta \vec{u}(\vec{x}, t) + \vec{a} \cdot \nabla \vec{u}(\vec{x}, t) = 0 \quad \forall (\vec{x}, t) \in \Omega \times [0, T], \quad (12)$$

$$\vec{u}(\vec{x}, t) = 0 \quad \forall \vec{x} \in \Gamma_- \quad 0 \leq t \leq T \quad (13)$$

$$\frac{\partial \vec{u}(\vec{x}, t)}{\partial \vec{n}} = 0 \quad \forall \vec{x} \in \partial\Omega/\Gamma_- \quad 0 \leq t \leq T \quad (14)$$

$$\vec{u}(\vec{x}, 0) = \vec{u}_0(\vec{x}) \quad \forall \vec{x} \in \Omega \quad (15)$$

and prove that,

$$\frac{1}{2} \int_{\Omega} (\vec{u}(\vec{x}, T))^2 dx + \int_0^T \int_{\Omega} \|\nabla \vec{u}(\vec{x}, t)\|^2 dx dt + \int_0^T \int_{\Omega/\Gamma_-} \vec{a} \cdot \vec{n} \frac{\vec{u}(x)^2}{2} ds dt = \frac{1}{2} \int_{\Omega} (\vec{u}_0(\vec{x}))^2 dx. \quad (16)$$

then, the second term is,

$$\int_0^1 u'(x)u(x)dx = \int_0^1 \frac{1}{2}(u^2(x))'dx = \frac{1}{2}[(u^2(x))]_0^1 = \frac{1}{2}u^2(1)$$

so we get to the conclusion,

$$\int_0^1 ((u'(x))^2 + (u(x))^2)dx + \frac{1}{2}u^2(1) = \int_0^1 f(x)u(x)dx. \quad (1)$$

(b) Multiply (8) by $\vec{u}(\vec{x})$ and integrate in the domain Ω . Doing similar steps to the one did before we get,

$$\int_{\Omega} (\|\nabla \vec{u}(\vec{x})\|^2 + (\vec{u}(\vec{x}))^2)dx + \int_{\Omega/\Gamma_-} \vec{a} \cdot \vec{n} \frac{\vec{u}(\vec{x})^2}{2} ds = \int_{\Omega} \vec{f}(\vec{x})\vec{u}(\vec{x})dx. \quad (2)$$

(c) Multiply (12) by $\vec{u}(\vec{x}, t)$ and integrate in the domain Ω doing similar steps to the one did for point 1 we get,

$$\int_{\Omega} \frac{\partial \vec{u}}{\partial t}(\vec{x}, t)\vec{u}(\vec{x}, t)dx + \int_{\Omega} \|\nabla \vec{u}(\vec{u}, t)\|^2 dx + \int_{\Omega/\Gamma_-} \vec{a} \cdot \vec{n} \frac{\vec{u}(\vec{x})^2}{2} ds = \quad (3)$$

$$\int_{\Omega} \frac{1}{2} \frac{\partial \vec{u}^2}{\partial t}(\vec{x}, t)dx + \int_{\Omega} \|\nabla \vec{u}(\vec{x}, t)\|^2 dx + \int_{\Omega/\Gamma_-} \vec{a} \cdot \vec{n} \frac{\vec{u}(\vec{x})^2}{2} ds = 0. \quad (4)$$

by integrating in time we arrive to the final formulation,

$$\frac{1}{2} \int_{\Omega} (\vec{u}(\vec{x}, T))^2 dx + \int_0^T \int_{\Omega} \|\nabla \vec{u}(\vec{x}, t)\|^2 dx dt = \frac{1}{2} \int_{\Omega} (\vec{u}_0(\vec{x}))^2 dx - \int_0^T \int_{\Omega/\Gamma_-} \vec{a} \cdot \vec{n} \frac{\vec{u}(\vec{x})^2}{2} ds dt. \quad (5)$$