EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2025

Problem Sheet 4

September 29, 2025

Question 1

Graded exercise for group 1

Let $T > 0$. Let $y : [0, T] \to \mathbb{R}^d$ be the solution of the problem

$$y'(t) = f(t, y(t)), \qquad 0 \leqslant t \leqslant T, \tag{1}$$
$$y(0) = y_0, \tag{2}$$

where $y_0 \in \mathbb{R}^d$ and $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a Lipschitz function of constant L.

Let N be a positive integer, $h = T/N$, $t_n = nh, n = 0, 1, \dots, N$. Consider the forward Euler scheme

$$\frac{y_{n+1} - y_n}{h} = f(t_n, y_n), \quad n = 0, 1, \dots, N-1.$$

We define

$$y_h(t) = y_n \frac{t_{n+1} - t}{h} + y_{n+1} \frac{t - t_n}{h}, \qquad t^n \leqslant t \leqslant t^{n+1}, n = 0, 1, 2, ..., N-1.$$

(a) Prove that

$$y_h'(t) = f(t, y_h(t)) + \eta(t), \quad t_n \leqslant t \leqslant t_{n+1},$$

where $\eta : [0, T] \to \mathbb{R}^d$ is computable if y_n, y_{n+1} are known.

(b) We set $E(t) = y(t) - y_h(t)$. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|E(t)\|^2 \leqslant 4L \|E(t)\|^2 + \frac{\|\eta(t)\|^2}{2L}. \tag{3}$$

(c) Prove that

$$\|E(T)\|^2 \leqslant \int_0^T \frac{\|\eta(t)\|^2 e^{4L(T-t)}}{2L} \, \mathrm{d}t. \tag{4}$$

Remark : the right hand side of (4) can be used to obtain an upper bound of the error $E(T)$ once the y_n are computed, $n = 1, \dots, N$.

Question 2

Implement the adaptive algorithm that was introduced in the lecture and apply it to ordinary differential equation given by

$$\begin{cases} \dot{y}(t) = -50\left(y(t) - cos(t)\right), & 0 < t \leqslant T, \\ y(0) = 0.15, \end{cases} \tag{5}$$

with $T = 7$. Compute y_n with the Runge method and $\hat{y}_n$ with the Forward Euler scheme.

Question 3

Prove the following Lemma of the lecture. Let y be the solution to the ODE

$$\begin{cases} y'(t) = f(t, y(t)), & t_0 \leqslant t \leqslant T, \\ y(t_0) = y_0, \end{cases}$$

and y_n a RK method given by

$$
\begin{aligned}
&y_{n+1} = y_n + h_n \Phi(t_n, y_n, h_n), \\
&t_n = t_{n-1} + h_n, \qquad n = 0, 1, 2, ..., N,
\end{aligned}
$$

where $\Phi(t, z, h)$ is defined as in the lecture.

If $f(t, z)$ satisfies a Lipschitz condition, that is there exists $L > 0$ such that

$$\|f(t, z_1) - f(t, z_2)\| \leqslant L\|z_1 - z_2\|, \qquad \forall (t, z_1), (t, z_2) \text{ in a neighbourhood of } (t, y(t)),$$

then there exists $\Lambda > 0$ such that

$$\|\Phi(t, z_1, h) - \Phi(t, z_2, h)\| \leqslant \Lambda\|z_1 - z_2\|, \qquad \forall t \leqslant T, 0 < h \leqslant h_{max}.$$

EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2025

Answer Key 4

September 29, 2025

Question 1

We have

$$y_h'(t) = \frac{y_{n+1} - y_n}{h} = f(t_n, y_n) = f(t, y_h(t)) + \eta(t),$$

where $\eta(t) = f(t_n, y_n) - f(t, y_h(t))$. Now, setting $E(t) = y(t) - y_h(t)$, we have :

$$E'(t) = f(t, y(t)) - f(t, y_h(t)) - \eta(t). \qquad (\star)$$

The Lipschitz condition on the function f yields $\|f(t, y(t)) - f(t, y_h(t))\| \leqslant L\|y_n - y_h(t)\| = L\|E(t)\|$.

By taking the scalar product of $(\star)$ with $E(t)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E(t)\|^2 = E'(t)\cdot E(t) &= (f(t, y(t)) - f(t, y_h(t)))\cdot E(t) - \eta(t)\cdot E(t) \\ &\leqslant \|f(t, y(t)) - f(t, y_h(t))\|\,\|E(t)\| + \|E(t)\|\,\|\eta(t)\| \\ &\leqslant L\|E(t)\|^2 + \|E(t)\|\,\|\eta(t)\| \\ &\leqslant L\|E(t)\|^2 + L\|E(t)\|^2 + \frac{1}{4L}\|\eta(t)\|^2,\end{aligned}$$

where the Young's inequality for the last upper bound. Hence we proved (3).

Let us multiply (3) by e^{-4LT}, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|E(t)\|^2 e^{-4Lt}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\|E(t)\|^2 e^{-4Lt} - 4Le^{-4Lt}\|E(t)\|^2 \leqslant \frac{\|\eta(t)\|^2}{2L}e^{-4Lt}.$$

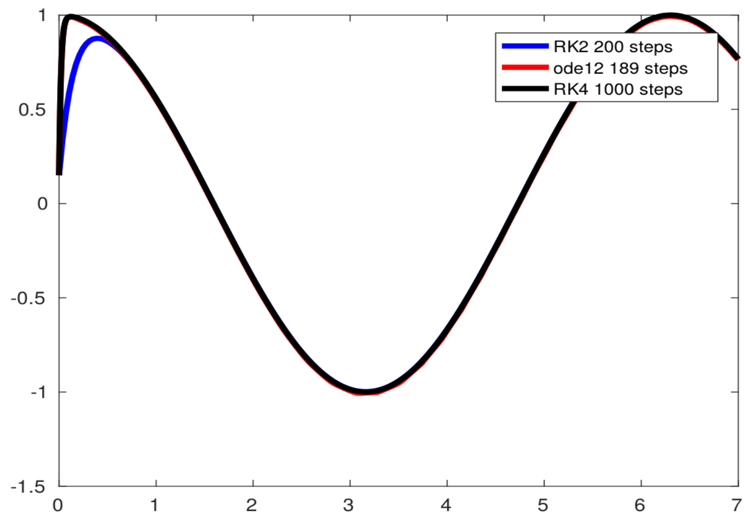
Integrating the results from $t = 0$ to $t = T$ yields

$$\|E(T)\|^2 e^{-4LT} \leqslant \|E(0)\|^2 + \int_0^T \frac{\|\eta(t)\|^2}{2L}e^{-4Lt}\,\mathrm{d}t.$$

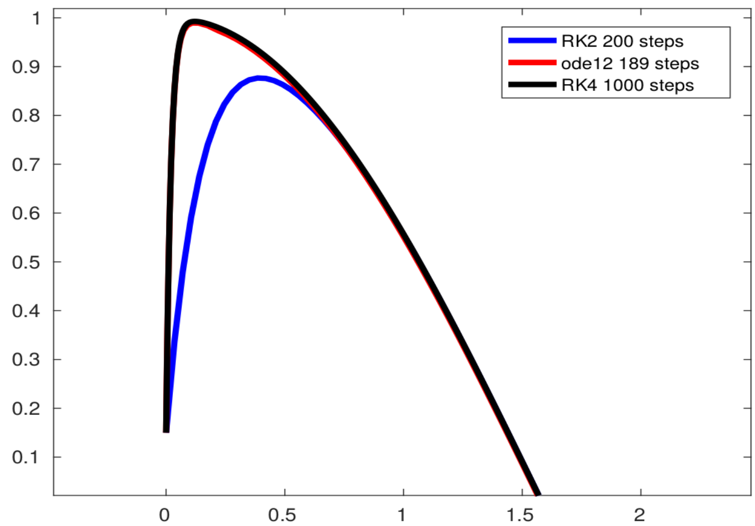
Since $E(0) = \vec{0}$, multiplying the result by e^{4LT} yields (4).

Question 2

For $T = 7$ and $y_0 = 0.15$, we have plotted the approximated solution to the given ODE using the Runge (RK2) method with a constant time step ($dt = 0.035$)and the adaptive algorithm with a tolerance $= 0.01$. We tend to compare the two methods by plotting the approximated solution using RK4 method with a high number of steps (1000) which is considered as very close to the exact solution.



Whe can observe that the adaptive algorithm with 189 steps is much more precise than Runge with 200 steps.



Question 3

We recall that $\Phi(t,z,h) = \sum_{i=1}^{s} b_i k_i(t,z,h)$ and $k_i(t,z,h) = f(t + c_i h, z + h(a_{i1}k_1(t,z,h) + ... + a_{ii-1}k_{i-1}(t,z,h)))$. We have

$$\|k_1(t,z_1,h) - k_1(t,z_2,h)\| = \|f(t+c_1h, z_1) - f(t+c_1h,z_2)\| \leqslant L\|z_1 - z_2\|,$$

$$\begin{aligned}\|k_2(t,z_1,h) - k_2(t,z_2,h)\| &= \|f(t+c_2h, z_1 + ha_{21}k_1(t,z_1,h)) - f(t+c_2h, z_2 + ha_{21}k_1(t,z_2,h))\| \\ &\leqslant L\|z_1 - z_2 + ha_{21}(k_1(t,z_1,h) - k_1(t,z_2,h))\| \leqslant L(1 + h|a_{21}L|)\|z_1 - z_2\|\end{aligned}$$

and so on. So we can conclude that

$$\begin{aligned}\|\Phi(t,z_1,h) - \Phi(t,z_2,h)\| &\leqslant \sum_{i=1}^{s} |b_i|\|k_i(t,z_1,h) - k_i(t,z_2,h)\| \\ &\leqslant L\left(\sum_{i=1}^{s}|b_i| + hL\sum_{i,j=1}^{s}|b_i a_{ij}| + h^2L^2\sum_{i,j,l=1}^{s}|b_i a_{ij} a_{jl}| + ...\right)\|z_1 - z_2\| \\ &\leqslant L\left(\sum_{i=1}^{s}|b_i| + h_{max}L\sum_{i,j=1}^{s}|b_i a_{ij}| + h_{max}^2L^2\sum_{i,j,l=1}^{s}|b_i a_{ij} a_{jl}| + ...\right)\|z_1 - z_2\|.\end{aligned}$$

Thus the result is proved with

$$\Lambda = L\left(\sum_{i=1}^{s}|b_i| + h_{max}L\sum_{i,j=1}^{s}|b_i a_{ij}| + h_{max}^2 L^2 \sum_{i,j,l=1}^{s}|b_i a_{ij} a_{jl}| + ...\right).$$