EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2025

Problem Sheet 3

September 22, 2025

Question 1

Consider an s-stages implicit Runge-Kutta method for solving $y'(t) = f(t, y(t))$, $y(t_0) = y_0 \in \mathbb{R}^d$

$$k_i = f(t_0 + c_i h, y_0 + h \sum_{j=1}^{s} a_{ij} k_j) \quad i = 1, ..., s, \tag{1}$$

$$y_1 = y_0 + h \sum_{i=1}^{s} b_i k_i$$

Here $c_i, b_i, a_{ij} \in \mathbb{R}$ and satisfy $c_1 = 0$, $c_i = \sum_{j=1}^{s} a_{ij}$.

Assume $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous (with constant L and norm $|| \cdot ||$) with respect to the second variable. Prove that there exists a unique solution of (1) if $h < \frac{1}{L \max\limits_{1 \leqslant i \leqslant s} \sum_{j=1}^{s} |a_{ij}|}$.

Indication : consider the fixed point iteration

$$k_i^{(m+1)} = f(t_0 + c_i h, y_0 + h \sum_{j=1}^{s} a_{ij} k_j^{(m)}) \quad m = 0, 1, 2, ... \tag{2}$$

and use Banach fixed point theorem.

Question 2

Graded exercise for group 3

Finish the proof of Theorem 1 to obtain the third order conditions.

Question 3

Consider the ordinary differential equation given by

$$\begin{cases} \dot{y}(t) = \lambda y(t), & 0 < t \leqslant T, \\ y(0) = y_0, \end{cases} \tag{3}$$

with $\lambda < 0$. Let N be a positive integer, let $h = \frac{T}{N}$ be the time step and $t_n = nh$ where $n = 0, 1, ..., N$. Consider now an order 4 RK scheme with 4 stages to approximate (3). Let

$$p_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}.$$

- Prove that $y_N = (p_4(\lambda h))^N y_0$.
- Prove that there exists $C > 0$ such that $\forall \lambda < 0, \forall h > 0$ such that $|p_4(\lambda h)| \leqslant 1$, $\forall T > 0$ and $\forall y_0 \in \mathbb{R}$,

$$|y(t_N) - y_N| \leqslant C |\lambda|^5 T h^4 |y_0|.$$

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Answer Key 3

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Question 1

Equation (2) can be written in compact form as

$$K^{(m+1)} = F(K^{(m)}),$$

with

$$K^{(m)} = (k_1^{(m)}, \dots, k_s^{(m)})^T, \qquad F_i(K) = f\left(t_0 + c_i h,\, y_0 + h \sum_{j=1}^{s} a_{ij} k_j\right),$$

for all $K = (k_1, \dots, k_s)^T \in \mathbb{R}^{ds}$ (recall $k_i \in \mathbb{R}^d$).
Now take $K = (z_1, \dots, z_s)^T$ and $\tilde{K} = (\tilde{z}_1, \dots, \tilde{z}_s)^T \in \mathbb{R}^{ds}$. By the Lipschitz continuity of f in the second variable,

$$\begin{aligned} \|F_i(K) - F_i(\tilde{K})\| &\leqslant Lh \left\| \sum_{j=1}^{s} a_{ij}(z_j - \tilde{z}_j) \right\| \\ &\leqslant Lh \sum_{j=1}^{s} |a_{ij}| \, \|z_j - \tilde{z}_j\|. \end{aligned}$$

Define the norm

$$\|K\| := \max_{1 \leqslant i \leqslant s} \|z_i\|.$$

Then for every j, $\|z_j - \tilde{z}_j\| \leqslant \|K - \tilde{K}\|$. Hence,

$$\|F_i(K) - F_i(\tilde{K})\| \leqslant Lh \Big(\sum_{j=1}^{s} |a_{ij}| \Big) \|K - \tilde{K}\|.$$

Taking the maximum over $i = 1, \dots, s$,

$$\begin{aligned} \|F(K) - F(\tilde{K})\| &= \max_{1 \leqslant i \leqslant s} \|F_i(K) - F_i(\tilde{K})\| \\ &\leqslant \Big(Lh \max_{1 \leqslant i \leqslant s} \sum_{j=1}^{s} |a_{ij}| \Big) \|K - \tilde{K}\|. \end{aligned}$$

Therefore, F is a contraction whenever

$$hL \max_{1 \leqslant i \leqslant s} \sum_{j=1}^{s} |a_{ij}| < 1.$$

By the Banach fixed point theorem, the iteration $K^{(m+1)} = F(K^{(m)})$ converges to the unique fixed point K^*, which is exactly the solution of (2).

Question 2

In the formalism introduced during the lecture, we have that

$$y(t_0 + h) = y_0 + h f_0 + \frac{h^2}{2} f_0' f_0 + \frac{h^3}{6} (f_0''(f_0, f_0) + f_0' f_0' f_0) + O(h^4).$$

We denote the numerical solution

$$y_1(h) = y_0 + \underbrace{h\sum_{i=1}^{3} b_i k_i(h)}_{\Phi(h)}.$$

We have the following expansion

$$y_1(h) = y_0 + \Phi(0) + h\Phi'(0) + \frac{h^2}{2}\Phi''(0) + \frac{h^3}{6}\Phi'''(0) + O(h^4).$$

To have a method of order 3, we need to derive an expression for $\Phi'''(0)$,

$$\Phi'''(h) = 3\sum_{i=1}^{s} b_i k_i''(h) + h\sum_{i=1}^{s} b_i k_i'''(h).$$

choosing h equal zero, implies that we have to derive an expression for $k_i''(0)$.
Again from lectur formalism $\psi = h\sum_{j=1}^{s} a_{ij}k_j(h)$,

$$k_i''(h) = f''(y_0+\psi_i(h))(\sum_{j=1}^{s} a_{ij}k_j(h) + h\sum_{j=1}^{s} a_{ij}k_j'(h))^2 + f'(y_0+\psi_i(h))(2\sum_{j=1}^{s} a_{ij}k_j'(h) + h\sum_{j=1}^{s} a_{ij}k_j''(h)),$$

which leads to,

$$k_i''(0) = f''(y_0)(f(y_0), f(y_0))\sum_{j,k=1}^{s} a_{ij}a_{ik} + 2f'(y_0)f'(y_0)f(y_0)\sum_{j,k=1}^{s} a_{ij}a_{jk}.$$

Then we have finished, it is enough to put all the elements togheter and evaluating $|y(t_0+h) - y_1|$.

Question 3

The exact solution of the ODE is :

$$y(t_N) = y_0 e^{\lambda N h}.$$

A 4 stages RK method applied to (3) yields :

$$\begin{aligned}
k_1 &= \lambda y_0\\
k_2 &= \lambda y_0 + h\lambda^2 a_{21} y_0\\
k_3 &= \lambda y_0 + h(a_{31}+a_{32})\lambda^2 y_0 + h^2 a_{32}a_{21}\lambda^3 y_0\\
k_4 &= \lambda y_0 + h(a_{41}+a_{42}+a_{43})\lambda^2 y_0 + h^2(a_{42}a_{21}+a_{43}a_{31}+a_{43}a_{32})\lambda^3 y_0 + h^3 a_{43}a_{32}a_{21}\lambda^4 y_0
\end{aligned}$$

and

$$y_1 = y_0 + h(b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4).$$

Using explicit expressions of the k_i and the 4-th order conditions, we have that

$$y_1 = p_4(\lambda h) y_0.$$

By induction, we get

$$y_N = (p_4(\lambda h))^N y_0.$$

Then

$$y(t_N) - y_N = y_0\left(e^{h\lambda N} - (p_4(\lambda h))^N\right) = y_0\left(e^{\lambda h} - p_4(\lambda h)\right)\left(e^{\lambda h(N-1)} + e^{\lambda h(N-2)}p_4(\lambda h) + ... + (p_4(\lambda h))^{N-1}\right).$$

Assuming $p_4(\lambda h) \leqslant 1$ and since $|e^x - p_4(x)| \leqslant \frac{1}{5!}x^5 \ \forall x > 0$, we have

$$|y(t_N) - y_N| \leqslant \frac{1}{5!}|\lambda|^5 h^5 N|y_0|.$$

Below, we plot the stability domain of the method in the complex plane, i.e. for $\lambda \in \mathbb{C}$ the set S_{RK} defined by

$$S_{RK} = \{z = h\lambda : |p_4(z)| \leqslant 1\}.$$

We compare it to the stability domain of the forward Euler scheme

$$S_E = \{z = h\lambda : |1+z| \leqslant 1\}.$$

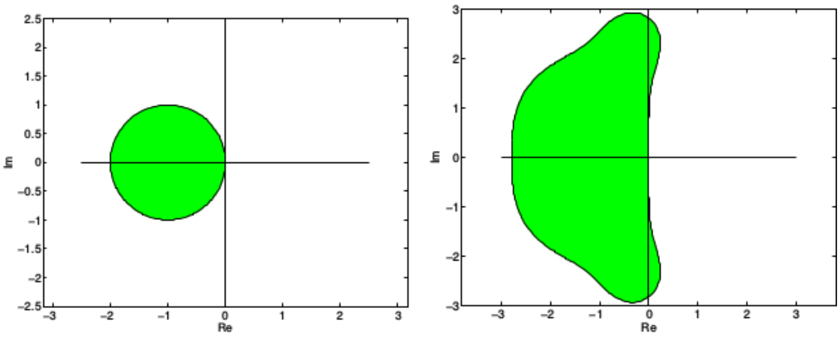

Figure 1 – Stability domain of the forward Euler scheme (left) and a 4-stages RK method of fourth order (right).