Problem Sheet 2

September 15, 2025

Question 1

Let us write a general s stages RK method for (2) as

$$\begin{aligned} K_1 &= F(Y(0)) \\ K_2 &= F(Y(0) + ha_{21}K_1) \\ K_3 &= F(Y(0) + h(a_{31}K_1 + a_{32}K_2)) \\ &\vdots \\ K_s &= F(Y(0) + h(a_{s1}K_1 + a_{s2}K_2 + ... + a_{ss-1}K_{s-1}) \\ Y_1 &= Y(0) + h(b_1K_1 + ... + b_sK_s) \end{aligned}$$

By definition of F, we have that

$$K_1 = \begin{pmatrix} 1 \\ f(t_0, y_0) \end{pmatrix}.$$

We set $k_1 = f(t_0, y_0)$ and we have

$$K_1 = \begin{pmatrix} 1 \\ k_1 \end{pmatrix}.$$

Then, we have

$$Y_0 + ha_{21}K_1 = \begin{pmatrix} t_0 + ha_{21} \\ y_0 + ha_{21}k_1 \end{pmatrix}.$$

Thus

$$K_2 = \begin{pmatrix} 1 \\ f(t_0 + ha_{21}, y_0 + ha_{21}k_1) \end{pmatrix}.$$

Reproducing this process until the $s-th$ stages, we find that

$$K_s = \begin{pmatrix} 1 \\ k_s \end{pmatrix}$$

with $k_s = f(t_0 + h(a_{s1} + a_{s2} + ... + a_{ss-1}), y_0 + h(a_{s1}k_1 + ... + a_{ss-1}k_{s-1}))$.
Therefore we have that

$$Y_1 = \begin{pmatrix} t_0 \\ y_0 \end{pmatrix} + h\left(b_1\begin{pmatrix} 1 \\ k_1 \end{pmatrix} + b_2\begin{pmatrix} 1 \\ k_2 \end{pmatrix} + ... + b_s\begin{pmatrix} 1 \\ k_2 \end{pmatrix}\right) = \begin{pmatrix} t_0 + h(b_1 + ... + b_s) \\ y_0 + h(b_1k_1 + ... + b_sk_s) \end{pmatrix}.$$

Thus, this is a s stages RK method provided

$$c_i = \sum_{j=1}^{i-1} a_{ij}, \quad 1 = \sum_{i=1}^{s} b_i.$$

Question 2

The exact solution of the ODE is :

$$y(t_N) = y_0 e^{\lambda N h}.$$

A 2 stages RK method applied to (3) yields :

$$k_1 = \lambda y_0$$
$$k_2 = \lambda y_0 + h\lambda^2 a_{21} y_0$$

and
$$y_1 = y_0 + h(b_1 k_1 + b_2 k_2).$$
From the thoery we know that $\sum_{i=1}^{2} b_i = 1$, also $\sum_{j=1}^{1} a_{2j} = c_2$ and $b_2 c_2 = \frac{1}{2}$. This lead to :
$$y_1 = y_0 \left(1 + h\lambda + \frac{(h\lambda)^2}{2}\right).$$
Finally,
$$y_1 = p_2(\lambda h) y_0.$$
By induction, we get
$$y_N = (p_2(\lambda h))^N y_0.$$
Then
$$y(t_N) - y_N = y_0 \left(e^{h\lambda N} - (p_2(\lambda h))^N\right) = y_0 \left(e^{\lambda h} - p_2(\lambda h)\right) \left(e^{\lambda h(N-1)} + e^{\lambda h(N-2)} p_2(\lambda h) + ... + (p_2(\lambda h))^{N-1}\right).$$
Assuming $p_2(\lambda h) \leqslant 1$ and since $|e^x - p_2(x)| \leqslant \frac{1}{3!}x^3 \; \forall x > 0$, we have
$$|y(t_N) - y_N| \leqslant \frac{1}{3!}|\lambda|^3 h^3 N |y_0|.$$
In the real domain, we can check that $|p_2(x)| < 1$. The inequality $|1 + x + \frac{x^2}{2}| < 1$ is equivalent to :
$$-1 < 1 + x + \frac{x^2}{2} < 1 \tag{1}$$
This gives two separate inequalities to solve : $1 + x + \frac{x^2}{2} > -1$ and $x\left(1 + \frac{x}{2}\right) < 0$.
Concerning the first one, the discriminant is $\Delta = 4 - 16 = -12 < 0$. Since it is negative and the coefficient of x^2 is positive, this quadratic is always positive. So this inequality is satisfied for all real x.

Moving to the second one, we have $x(2 + x) < 0$. So, we need to find where the product is negative. The roots are $x = 0$ and $x = -2$.

Using a sign chart :
- For $x < -2$: $x < 0$ and $(2 + x) < 0$, so the product is positive
- For $-2 < x < 0$: $x < 0$ and $(2 + x) > 0$, so the product is negative
- For $x > 0$: $x > 0$ and $(2 + x) > 0$, so the product is positive

Therefore, $x(2 + x) < 0$ when $-2 < x < 0$. Substituting $x = h\lambda$, we get that $h \leqslant -\frac{2}{\lambda}$.