

# Problem Sheet 1

September 09, 2025

## Question 1

- We take the scalar product with  $\vec{u}(t)$  and obtain  $\frac{1}{2} \frac{d}{dt} \|\vec{u}(t)\|^2 + \vec{u}(t)^T A \vec{u}(t) = \vec{u}(t)^T \vec{f}(t)$ . We have  $\vec{u}^T(t) A \vec{u}(t) \geq \lambda_{\min} \|\vec{u}(t)\|^2$ . Using Cauchy-Schwarz and Young inequality ( $ab \leq \frac{\lambda}{2} a^2 + \frac{1}{2\lambda} b^2$ ) we find

$$\frac{1}{2} \frac{d}{dt} \|\vec{u}(t)\|^2 + \lambda_{\min} \|\vec{u}(t)\|^2 \leq \frac{\lambda_{\min}}{2} \|\vec{u}(t)\|^2 + \frac{1}{2\lambda_{\min}} \|\vec{f}(t)\|^2,$$

which yields the result integrating from  $t = 0$  to  $t = T$ .

- From Taylor expansion we find  $\vec{u}(t_n) = \vec{u}(t_{n+1}) - h\vec{u}'(t_{n+1}) + \frac{h^2}{2}\vec{u}''(s_n)$ , with  $t_n < s_n < t_{n+1}$ . This yields

$$\frac{\vec{u}(t_{n+1}) - \vec{u}(t_n)}{h} + A\vec{u}(t_{n+1}) = \vec{f}(t_{n+1}) - \frac{h}{2}\vec{u}''(s_n),$$

hence  $\vec{r}^{n+1} = -\frac{h}{2}\vec{u}''(s_n)$  and  $\|\vec{r}^{n+1}\| \leq \frac{1}{2}h \max_{0 \leq t \leq T} \|\vec{u}''(t)\|$ .

- Subtracting (2) and (3) and taking the scalar product with  $\vec{e}^{n+1}$  we find  $\frac{1}{h} (\vec{e}^{n+1} - \vec{e}^n)^T \vec{e}^{n+1} + (\vec{e}^{n+1})^T A \vec{e}^{n+1} = (\vec{e}^{n+1})^T \vec{r}^{n+1}$ . Thus,  $\frac{1}{2h} (\|\vec{e}^{n+1}\|^2 - \|\vec{e}^n\|^2 + \|\vec{e}^{n+1} - \vec{e}^n\|^2) + \lambda_{\min} \|\vec{e}^{n+1}\|^2 \leq \frac{\lambda_{\min}}{2} \|\vec{e}^{n+1}\|^2 + \frac{1}{2\lambda_{\min}} \|\vec{r}^{n+1}\|^2$ , which yields the result.
- Summing from  $n = 0$  to  $N - 1$  and since  $\vec{e}^0 = 0$  we find

$$\frac{1}{h} \|\vec{e}^N\|^2 + \lambda_{\min} \sum_{n=0}^{N-1} \|\vec{e}^{n+1}\|^2 \leq \sum_{n=0}^{N-1} \frac{1}{\lambda_{\min}} \|\vec{r}^{n+1}\|^2 \leq \frac{NC^2 h^2}{\lambda_{\min}}.$$

From  $h = \frac{T}{N}$  we obtain our result.

## Question 2

- (a) We have  $\vec{u}(t_{n+1}) = \vec{u}(t_n) + h\vec{u}'(t_n) + \frac{1}{2}h^2\vec{u}''(s_n)$ ,  $t_n < s_n < t_{n+1}$ . Thus

$$\frac{\vec{u}(t_{n+1}) - \vec{u}(t_n)}{h} + A\vec{u}(t_n) = \vec{f}(t_n) + \vec{r}^n,$$

with  $\vec{r}^n = \frac{1}{2}h\vec{u}''(s_n)$  and  $\|\vec{r}^n\| \leq \frac{1}{2}h \max_{0 \leq t \leq T} \|\vec{u}''(t)\|$ . Defining  $\vec{e}^n = \vec{u}(t_n) - \vec{u}^n$ , we have

$$\frac{\vec{e}^{n+1} - \vec{e}^n}{h} + A\vec{e}^n = \vec{r}^n.$$

Multiplying the result by  $\vec{e}^{n+1}$  it yields

$$\frac{1}{h} (\vec{e}^{n+1} - \vec{e}^n)^T \vec{e}^{n+1} + (\vec{e}^{n+1})^T A \vec{e}^n = (\vec{e}^{n+1})^T \vec{r}^n,$$

which yields the result, using  $a = (a - b) + b$  and  $(a - b)a = \frac{1}{2}(a^2 - b^2 + (a - b)^2)$ .

- (b) Defining the scalar product  $\langle \vec{u}, \vec{v} \rangle_A = \vec{u}^T A \vec{v}$ , we use the Cauchy-Schwarz and Young inequality to find  $\langle \vec{u}, \vec{v} \rangle_A \leq \|\vec{u}\|_A \|\vec{v}\|_A \leq \frac{1}{2} \|\vec{u}\|_A^2 + \frac{1}{2} \|\vec{v}\|_A^2$ . This yields the result choosing  $\vec{u} = \vec{e}^{n+1}$  and  $\vec{v} = \vec{e}^{n+1} - \vec{e}^n$ .

(c) From (a) and (b), we have

$$\frac{1}{2h} \left( \|\bar{e}^{n+1}\|^2 - \|\bar{e}^n\|^2 + \|\bar{e}^{n+1} - \bar{e}^n\|^2 \right) + \frac{1}{2} (\bar{e}^{n+1})^T A \bar{e}^{n+1} \leq (\bar{e}^{n+1})^T \bar{r}^n + \frac{1}{2} (\bar{e}^{n+1} - \bar{e}^n)^T A (\bar{e}^{n+1} - \bar{e}^n).$$

The result is obtained since  $\lambda_{\min} \|\bar{v}\|^2 \leq \bar{v}^T A \bar{v} \leq \lambda_{\max} \|\bar{v}\|^2$ ,  $\forall \bar{v} \in \mathbb{R}^d$  (from decomposition into orthonormal basis of eigenvectors of  $A$ ).

(d) We have

$$\begin{aligned} \left( \frac{1}{2h} + \frac{\lambda_{\min}}{2} \right) \|\bar{e}^{n+1}\|^2 + \left( \frac{1}{2h} - \frac{\lambda_{\max}}{2} \right) \|\bar{e}^{n+1} - \bar{e}^n\|^2 &\leq \frac{1}{2h} \|\bar{e}^n\|^2 + \|\bar{e}^{n+1}\| \|\bar{r}^n\| \\ &\leq \frac{1}{2h} \|\bar{e}^n\|^2 + \frac{\lambda_{\min}}{2} \|\bar{e}^{n+1}\|^2 + \frac{1}{2\lambda_{\min}} \|\bar{r}^n\|^2. \end{aligned}$$

If  $\frac{1}{2h} - \frac{\lambda_{\max}}{2} \geq 0$ , this yields  $\|\bar{e}^{n+1}\|^2 \leq \|\bar{e}^n\|^2 + \frac{h}{\lambda_{\min}} \|\bar{r}^n\|^2 \leq \|\bar{e}^n\|^2 + \frac{C^2 h^3}{\lambda_{\min}}$ . Summing from  $n = 0$  to  $n = N - 1$  and using  $T = Nh$  yields the result.

### Question 3

We have from (4) :

$$\frac{1}{2} \frac{d}{dt} \|\bar{x}'(t)\|^2 = \bar{x}''(t)^T \bar{x}'(t) = \frac{-\bar{x}(t)^T \bar{x}''}{\|\bar{x}(t)\|^3} = \frac{d}{dt} \frac{1}{\|\bar{x}(t)\|}.$$

Now let  $\bar{x}'(t) = \bar{v}(t)$ , (1) is equivalent to  $\bar{u}'(t) = f(\bar{u}(t))$ , with  $\bar{u}(t) = \begin{pmatrix} \bar{x}(t) \\ \bar{v}(t) \end{pmatrix}$  and  $f(\bar{x}, \bar{v}) = \begin{pmatrix} \bar{v} \\ -\frac{\bar{v}\bar{x}}{\|\bar{x}\|^3} \end{pmatrix}$ . Let  $N$  be a positive integer,  $h = \frac{T}{N}$ ,  $t_n = nh$ ,  $n = 0, 1, \dots, N$ . We compute  $\bar{u}^{n+1} \in \mathbb{R}^d$ ,  $n = 0, 1, \dots, N - 1$ , with the Euler explicit scheme :

$$\frac{\bar{u}^{n+1} - \bar{u}^n}{h} = \vec{f}(\bar{u}^n).$$

Let  $\bar{e}$  be the error at final time, the following results are obtained :

$h$	$\ \bar{e}\ $ explicit
0.0002	0.330
0.0001	0.169
0.00005	0.085
0.000025	0.042

which shows that the error is  $O(h)$ .

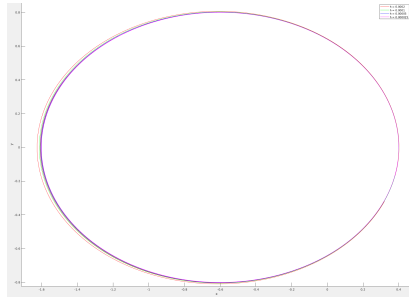


FIGURE 1 – Numerical solution with the Forward Euler method for the different  $h$ s.

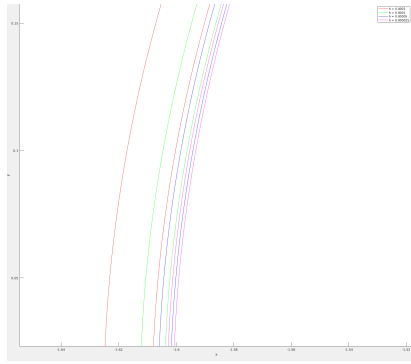


FIGURE 2 – Zoom of the left side of the previous plot.

## Question 4

We consider the ODE

$$y'(t) = -ay(t), \quad a > 0.$$

The implicit Euler discretization reads

$$\frac{y^{n+1} - y^n}{h} = -ay^{n+1},$$

which gives

$$y^{n+1} = \frac{y^n}{1 + ah}.$$

The analytical solution of the problem is

$$y(t) = y_0 e^{-at}.$$

The implicit Euler scheme is *A-stable*, i.e. it is stable for all step sizes

$$h > 0.$$

The method is of first order : the global error satisfies

$$\|y(t_n) - y^n\| = O(h).$$

### Solution Plot ( $T = 1$ )

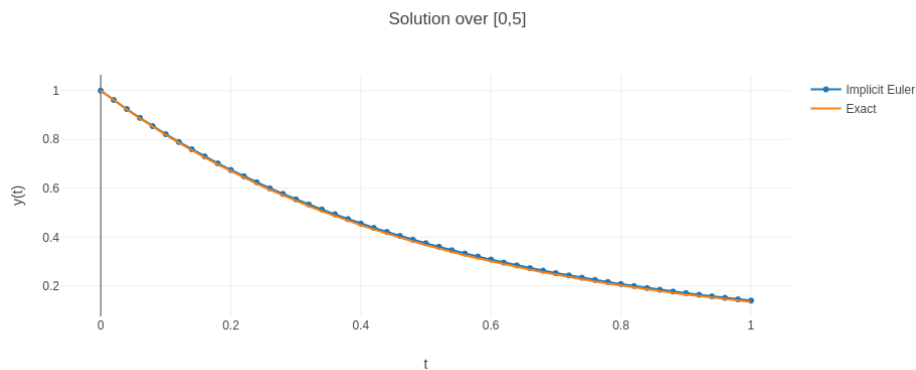
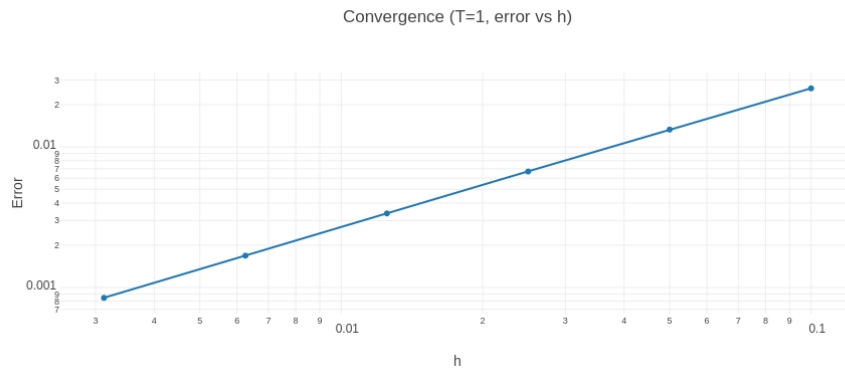


FIGURE 3 – Implicit Euler vs exact solution on  $[0, 1]$ .

### Convergence ( $T = 1$ )

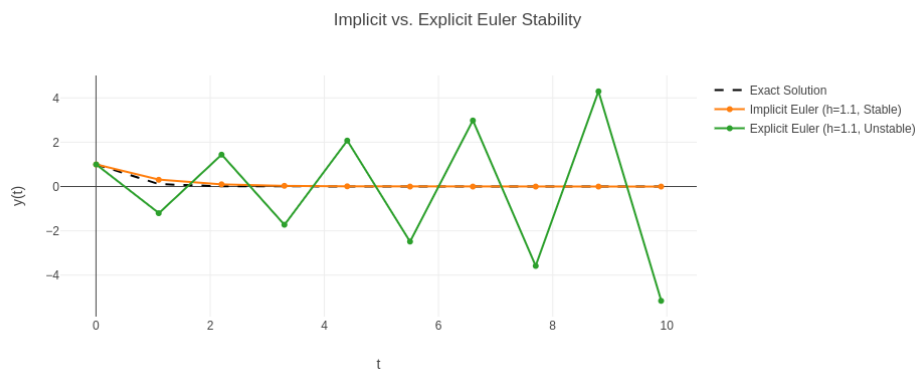
**Stability ( $T = 10$ )** This plot compares the behavior of Implicit and Explicit Euler methods using a large time step ( $h=1.1$ ) that violates the stability condition for Explicit Euler ( $h < 1.0$ ). Notice how the Implicit Euler solution remains stable (decays), while the Explicit Euler solution blows up.



**FIGURE 4** – Convergence plot : error vs step size  $h$  (log-log).

$N$	$h$	Error
10	1.00e-1	2.18e-3
20	5.00e-2	1.10e-3
40	2.50e-2	5.49e-4
80	1.25e-2	2.74e-4
160	6.25e-3	1.37e-4
320	3.13e-3	6.87e-5

**TABLE 1** – Convergence table : step size  $h$  and error at  $T = 1$ .



**FIGURE 5** – Stability plot :  $h=1.1$  and  $T=10$ .

## Question 5

We consider the system

$$y'(t) + Ay(t) = 0, \quad y(0) = y_0,$$

where  $A \in \mathbb{R}^{n \times n}$  is symmetric positive definite (SPD).

The implicit Euler discretization with step size  $h$  reads :

$$\frac{y^{n+1} - y^n}{h} + Ay^{n+1} = 0,$$

which gives

$$y^{n+1} = (I + hA)^{-1}y^n.$$

The exact solution is

$$y(t) = e^{-At}y_0.$$

Since  $A$  is SPD, all its eigenvalues are positive. Implicit Euler is  $A$ -stable, so the scheme is stable for all  $h > 0$ . The method is first order :

$$\|y(T) - y^N\| = \mathcal{O}(h).$$

**Convergence test ( $T = 1$ )** We compute the error at  $T = 1$  for different step sizes.

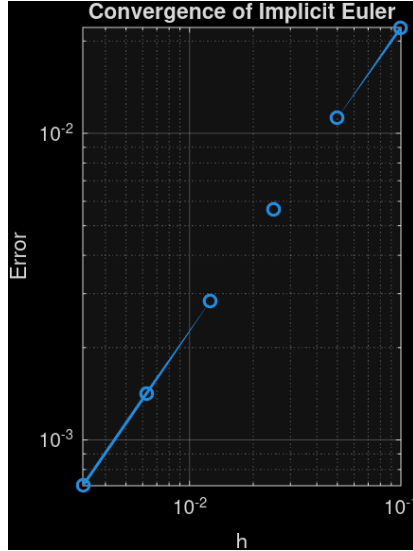


FIGURE 6 – Convergence plot : error vs step size  $h$  (log-log).

$N$	$h$	Error
10	0.1	0.022147
20	0.05	0.01121
40	0.025	0.0056391
80	0.0125	0.002828
160	0.00625	0.0014161
320	0.003125	0.00070857

### Stability test

To test the stability we take into consideration the Explicit Euler scheme. It is defined as  $\mathbf{y}_{n+1} = \mathbf{y}_n - hA\mathbf{y}_n = (I - hA)\mathbf{y}_n$ . The eigenvalues of the matrix  $(I - hA)$  are  $\nu_i = 1 - h\lambda_i$ . From the theory we know that for this method to be stable we need :

$$h_{critic} < \frac{2}{\lambda_{\max}(I - hA)}.$$

To best demonstrate the robustness of Implicit Euler, we choose a step size  $h$  that intentionally violates the stability condition for Explicit Euler.

By using  $h = 10\% * h_{critic}$ , we guarantee that the Implicit Euler method will produce a decaying (stable) solution, while the Explicit Euler method will produce a diverging (unstable) solution.

### Conclusion

- Implicit Euler is unconditionally stable for SPD matrices.
- The convergence rate is first order :  $\mathcal{O}(h)$ .
- Even for large step sizes, the scheme remains stable, though accuracy deteriorates.

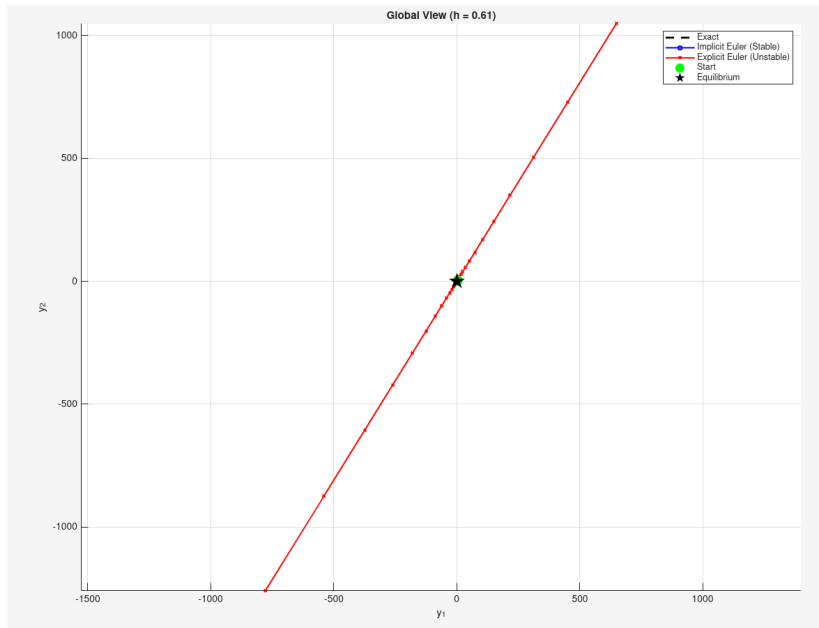


FIGURE 7 – Stability plot, solution  $y_1$  vs  $y_2$  over time.

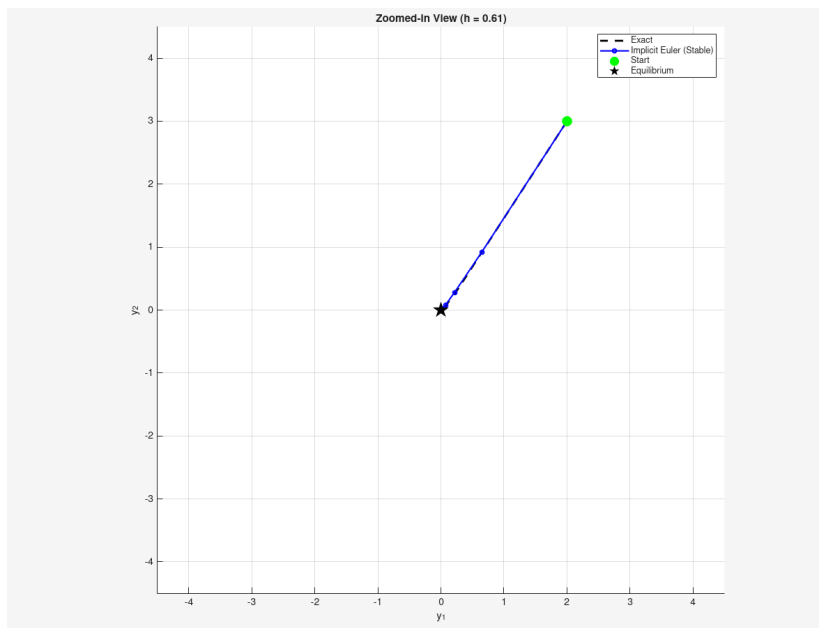


FIGURE 8 – Zoom of the Stability plot.