Problem Sheet 12

December 1, 2025

Question 1

Graded Exercise for Group 3

Let $u :]0,1[^2 \to \mathbb{R}$ be such that :

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,y,t) + \dfrac{\partial u}{\partial x}(x,y,t) = 0, \qquad (x,y) \in]0,1[^2, \qquad t > 0; \\ u(x,y,0) = 1, \qquad (x,y) \in]0,1[^2; \\ u(0,y,t) = 0, \qquad y \in]0,1[\qquad t > 0. \end{cases} \tag{1}$$

(a) What is the exact solution at $t = 0.5$?

(b) Fill the **trasp2d.m** file, run the code with $N = 99, 199, 399$ and $M = 60, 120, 240$ to check (eye) convergence.

(c) What happens whe $N = 99$ and $M = 50$?

Question 2

Given $u_0, v_0 : (0,1)^2 \to \mathbb{R}$, let u be the solution of the 2D wave equation with damping :

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,y,t) - \Delta u(x,y,t) + \dfrac{\partial u}{\partial t}(x,y,t) = 0, \qquad (x,y) \in (0,1)^2, \qquad t > 0; \\ u(x,y,0) = u_0(x,y), \qquad (x,y) \in (0,1)^2; \\ \dfrac{\partial u}{\partial t}(x,y,0) = v_0(x,y), \qquad (x,y) \in (0,1)^2; \\ u(x,y,t) = 0, \qquad (x,y) \in \partial(0,1)^2 \qquad t > 0. \end{cases} \tag{2}$$

Prove that for all $t > 0$,

$$\frac{d}{dt}\left(\int_0^1 \int_0^1 \left(\left(\frac{\partial u}{\partial t}(x,y,t)\right)^2 + ||\nabla u(x,y,t)||^2\right)\right) dxdy \leqslant 0$$

Hint : multiply the wave equation by $\frac{\partial u}{\partial t}$ and integrate by parts.

EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2025

Answer Key 12

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Question 1

(a) The exact solution is discontinuos, it is 0 for $t < 0.5$ and 1 for $t > 0.5$

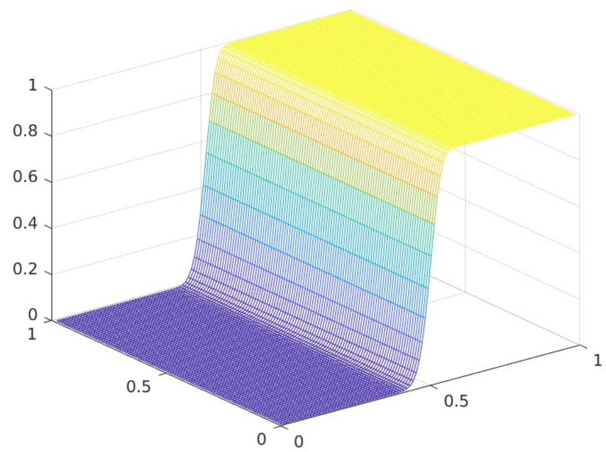

FIGURE 1 – Solution of the trasport problem for N=99 and M=60.

(b) The lines of the code are :

— Line 7 : E = sparse(2 :L,1 :L-1,1,L,L) ;

— Line 8 : A = kron(I,I-E)*(L+1) ;

— Line 9 : mat=speye(L*L,L*L)-tau*A ;

— Line 12 : u=mat*u ;

You can observe that letting $h \to 0$ and $\tau \to 0$ the solution at the discontinuity becomes steeper and steeper getting ever closer to the exact solution.

(c) For $N = 99$ and $M = 50$ the CFL number is equal to 1. This lead to have the exact solution since $h = \tau$ and there is a node in $t = 0.5$:

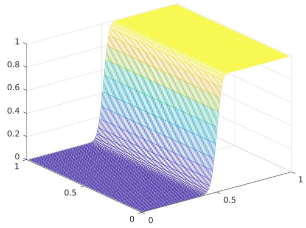

FIGURE 2 – Solution of the trasport problem for N=99 and M=60, CFL = 0.8333.

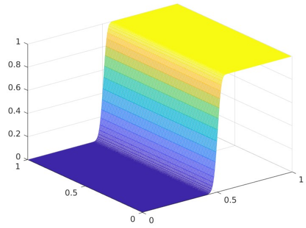

FIGURE 3 – Solution of the trasport problem for N=199 and M=120, CFL = 0.8333.

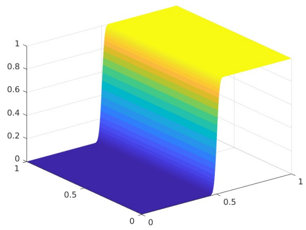

FIGURE 4 – Solution of the trasport problem for N=399 and M=240, CFL = 0.8333.

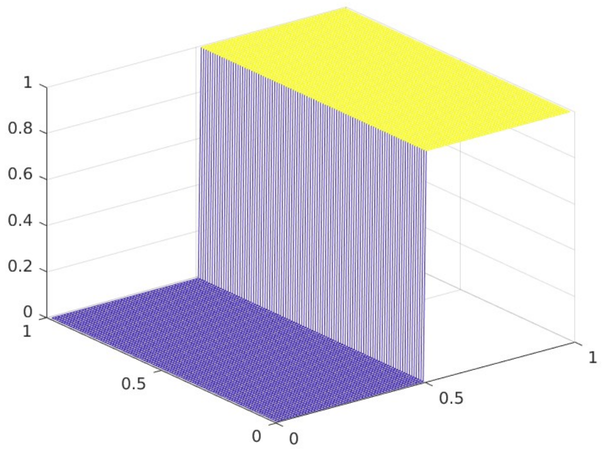

FIGURE 5 – Solution of the trasport problem for N=99 and M=50, CFL=1.

Question 2

Following the hint you can rewrite the first two terms as :

$$\int_0^1 \int_0^1 \left(\frac{\partial^2 u}{\partial t^2}(x,y,t) \frac{\partial u}{\partial t}(x,y,t) \right) dxdy = \frac{1}{2} \frac{d}{dt} \int_0^1 \int_0^1 \left(\frac{\partial u}{\partial t}(x,y,t) \right)^2 dxdy,$$

$$-\int_0^1 \int_0^1 \left(\Delta u(x,y,t) \frac{\partial u}{\partial t}(x,y,t) \right) dxdy = \int_0^1 \int_0^1 \left(\nabla u(x,y,t) \nabla \frac{\partial u}{\partial t}(x,y,t) \right) dxdy = \frac{1}{2} \frac{d}{dt} \int_0^1 \int_0^1 |\nabla u(x,y,t)|^2 dxdy.$$

Then, we can rewrite the equation as :

$$\frac{d}{dt} \left(\int_0^1 \int_0^1 \left(\left(\frac{\partial u}{\partial t}(x,y,t) \right)^2 + ||\nabla u(x,y,t)||^2 \right) \right) dxdy = -2 \int_0^1 \int_0^1 \left(\frac{\partial u}{\partial t}(x,y,t) \right)^2 dxdy \leqslant 0$$