Problem Sheet 11

November 24, 2025

Question 1

Implement the upwind and Lax-Wendroff schemes for the transport equation :

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + c_0 \dfrac{\partial u}{\partial x}(x,t) = 0, \qquad 0 < x < 1, \qquad 0 < t < 0.48, \qquad c_0 = \sqrt{2}; \\ u(0,t) = 0, \qquad t > 0; \\ u(x,0) = e^{-1000(x-0.2)^2}, \qquad 0 < x < 1. \end{cases} \tag{1}$$

Compare both schemes at time 0.48 when $h = 0.01$, $\Delta t = 0.006$ and check convergence.

Question 2

Graded Exercise for Group 2

Consider the Lax-Wendroff scheme for the transport equation :

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + c_0 \dfrac{\partial u}{\partial x}(x,t) = 0, \quad x \in \mathbb{R},\ t > 0,\ c_0 > 0; \\ u(x,0) = e^{imx} \qquad x \in \mathbb{R},\ m \in \mathbb{N}^*. \end{cases} \tag{2}$$

(a) Check that $u_j^n = u_j^0 \gamma^n$, with $\gamma = 1 - \alpha^2(1 - \cos(mh)) - i\alpha \sin(mh)$, $\alpha = \dfrac{c_0 \tau}{h}$.

(b) Check that $|\gamma|^2 = 1 + \alpha^2(\alpha^2 - 1)(1 - \cos(mh))^2$, so that $|\gamma| \leqslant 1$ when $\alpha \leqslant 1$.

Answer Key 11

November 24, 2025

Question 1

Numerical approximation of the advection equation for $N = 99$ and $M = 80$, so that $0.006 = \tau \leqslant \frac{h}{|c_0|} = 0.007$:

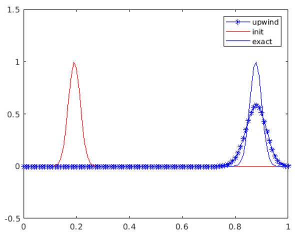

FIGURE 1 – Upwind scheme.

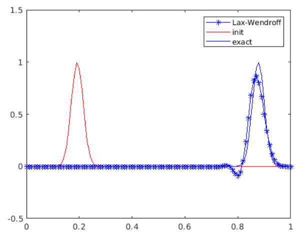

FIGURE 2 – Lax-Wendroff scheme.

For the Upwind scheme :

N	M	err_{inf}
49	40	$5.58 \cdot 10^{-1}$
99	80	$4.26 \cdot 10^{-1}$
199	160	$2.96 \cdot 10^{-1}$
399	320	$1.86 \cdot 10^{-1}$
799	640	$1.08 \cdot 10^{-1}$

For the Lax-Wendroff scheme :

N	M	err_{inf}
49	40	$3.96 \cdot 10^{-1}$
99	80	$2.23 \cdot 10^{-1}$
199	160	$9.09 \cdot 10^{-2}$
399	320	$2.45 \cdot 10^{-2}$
799	640	$6.13 \cdot 10^{-3}$

Question 2

(a) We show it by induction. The case $n = 0$ is trivially true. We assume that the property holds true for n and prove it for $n + 1$.
We have

$$\begin{aligned} u_j^{n+1} &= u_j^0\gamma^n - \frac{\alpha}{2}(u_{j+1}^0\gamma^n - u_{j-1}^0\gamma^n) + \frac{\tau^2 c_0^2}{2h^2}(u_{j-1}^0\gamma^n - 2u_j^0\gamma^n + u_{j+1}^0\gamma^n) \\ &= \gamma^n e^{imjh}(1 - \alpha\frac{e^{imh} - e^{-imh}}{2} - \alpha^2 + \alpha^2\frac{e^{imh} + e^{-imh}}{2} \\ &= e^{imjh}\gamma^n(1 - \alpha^2(1 - \cos(mh)) - i\alpha\sin(mh)) \\ &= e^{imjh}\gamma^n\gamma. \end{aligned} \tag{1}$$

(b)

$$\begin{aligned} |\gamma|^2 &= 1 - \alpha^2(1 - \cos(mh))^2 + \alpha^2\sin^2(mh) \\ &= 1 - 2\alpha^2(1 - \cos(mh)) + \alpha^4(1 - \cos(mh))^2 + \alpha^2(1 - \cos^2(mh)) \\ &= \alpha^4(1 - \cos(mh))^2 - \alpha^2(1 - \cos(mh))^2 + 1 \\ &= 1 + \alpha^2(\alpha^2 - 1)(1 - \cos(mh))^2. \end{aligned} \tag{2}$$