

Problem Sheet 10

November 17, 2025

Question 1

Use the file `heat.m` from last week exercitation to check the convergence results proved in the lecture for the implicit scheme when $T = 1$, $f(x, t) = 0$ and $u_0(x) = \sin(\pi x)$. (Try `heat(9, 10)`; `heat(19, 40)`; `heat(39, 160)`, ...)

Question 2

Implement the explicit scheme for the 1D heat equation when $f(x, t) = 0$ and $u_0(x) = \sin(\pi x)$.

(a) Choose $N = 9$, $\tau = \frac{h^2}{2} = 0.005$ and $M = 10$. Then multiply N by 2, M by 4, divide τ by 4, and so on ...

(b) Check stability : if $\tau < \frac{h^2}{2}$, then $\lim_{n \rightarrow \infty} \|u^n\|_\infty = 0$.

For instance, set $N = 19$, $\tau = \frac{h^2}{2} = 0.00125$, $M = 10^6$; $\|u^M\|$ should be close to 0. Then set $N = 19$, $\tau = \frac{h^2}{2} = 0.00126$, $M = 10^6$; $\|u^M\|$ should be extremely large!

Question 3

Graded Exercise for Group 1

Consider the parabolic advection-diffusion problem of Sheet 9, Question 2.

The scheme writes

$$\frac{\bar{u}^{n+1} - \bar{u}^n}{\tau} + A\bar{u}^{n+1} = \bar{\mathbf{1}},$$

where $A = \epsilon A_1 + A_2$,

$$A_1 = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & & -1 & 2 \end{pmatrix} \quad A_2 = \frac{1}{2h} \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & & 1 & 0 \end{pmatrix}.$$

We know that, for every $v \in \mathbb{R}^N$, $\lambda_1 v^T v \leq v^T A_1 v \leq \lambda_N v^T v$, where $\lambda_i = 2 \frac{1 - \cos(i\pi h)}{h^2}$, $i = 1, \dots, N$.

1. Prove that, for every $v \in \mathbb{R}^N$, $v^T A_2 v = 0$.

2. Using $(\bar{a} - \bar{b})^T \bar{a} = \frac{1}{2} (\|\bar{a}\|_2^2 - \|\bar{b}\|_2^2 + \|\bar{a} - \bar{b}\|_2^2)$, prove that

$$\|\bar{u}^{n+1}\|_2^2 + \tau \epsilon (\bar{u}^{n+1})^T A_1 \bar{u}^{n+1} \leq \frac{\tau \|\bar{\mathbf{1}}\|_2^2}{\lambda_1 \epsilon} + \|\bar{u}^n\|_2^2$$

so that

$$\|\bar{u}^M\|_2^2 + \tau \epsilon \sum_{n=0}^{M-1} (\bar{u}^{n+1})^T A_1 \bar{u}^{n+1} \leq T \frac{\|\bar{\mathbf{1}}\|_2^2}{\lambda_1 \epsilon}.$$

Question 4

Consider the 2d parabolic convection-diffusion problem ($\epsilon > 0$)

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t}(x_1, x_2, t) - \epsilon \frac{\partial^2 u}{\partial x^2}(x_1, x_2, t) - \frac{\partial u}{\partial x_1}(x_1, x_2, t) = 1, (x_1, x_2) \in]0, 1[^2, 0 < t \leq 10; \\ u(x_1, x_2, t) = 0, (x_1, x_2) \in \partial(]0, 1[^2), 0 < t \leq 10; \\ u(x_1, x_2, 0) = 0, (x_1, x_2) \in \partial(]0, 1[^2). \end{array} \right.$$

The matlab file `diffconv2dparab.m` implements a centered finite difference method for solving the above problem. Let L be the number of internal points in the $]0, 1[$ interval, let $h = \frac{1}{L+1}$ be the space step. Let M be the number of time steps $\tau = \frac{T}{M}$ the time step.

When running the file, approximation of $u(\frac{i}{L+1}, \frac{j}{L+1}, \frac{n}{M})$ are provided, $i, j = 1, \dots, L, n = 1, \dots, M$.

Fill the matlab file.

Provide an approximation of $\|u(\cdot, \cdot, 10)\|_{L^\infty(]0, 1[^2)}$ with three digits when $\epsilon = 0.1$.

Answer Key 10

November 17, 2025

Question 1

The L^∞ error converges as $\mathcal{O}(h^2 + \tau)$ (the error is divided by $2^2 = 4$ each time) whereas the L^2 error converges as $\mathcal{O}\left(\frac{h^2 + \tau}{h^{\frac{1}{2}}}\right)$ (the error is divided by $2^{\frac{3}{2}}$ each time).

N	M	err_{inf}	err_2
9	10	$2.03 \cdot 10^{-2}$	$4.54 \cdot 10^{-2}$
19	40	$5.24 \cdot 10^{-3}$	$1.66 \cdot 10^{-2}$
39	160	$1.32 \cdot 10^{-3}$	$5.90 \cdot 10^{-3}$
79	640	$3.31 \cdot 10^{-4}$	$2.09 \cdot 10^{-3}$

Question 2

(a) The L^∞ error converges as $\mathcal{O}(h^2 + \tau)$ (the error is divided by $2^2 = 4$ each time) whereas the L^2 error converges as $\mathcal{O}\left(\frac{h^2 + \tau}{h^{\frac{1}{2}}}\right)$ (the error is divided by $2^{\frac{3}{2}}$ each time).

N	M	err_{inf}	err_2
9	10	$5.07 \cdot 10^{-3}$	$1.33 \cdot 10^{-2}$
19	40	$1.25 \cdot 10^{-3}$	$3.94 \cdot 10^{-3}$
39	160	$3.10 \cdot 10^{-4}$	$1.39 \cdot 10^{-3}$
79	640	$7.74 \cdot 10^{-5}$	$4.90 \cdot 10^{-4}$

(b) For $N = 19$, $\tau = 0.00125$ and $M = 10^6$, $\|\bar{u}^M\|_\infty$ is approximately equal to zero. For $N = 19$, $\tau = 0.00126$ and $M = 10^6$ $u_i^M = Inf$.

Question 3

(a)

$$\begin{aligned} \bar{v}^T A_2 \bar{v} &= (v_1, v_2, \dots, v_N)^T (-v_2, v_1 - v_3, \dots, v_{N-2} - v_N, v_{N-1}) \\ &= -v_1 v_2 + v_1 v_2 - v_3 v_2 + \dots + v_{N-1} v_N \\ &= 0. \end{aligned}$$

Remark that, for every $B \in \mathbb{R}^{N \times N}$ satisfying $B^T = -B$, we have $\bar{v}^T B \bar{v} = 0 \forall \bar{v} \in \mathbb{R}^N$.

(b) Using the scheme that \bar{u} satisfies and the indication, we have

$$\begin{aligned} \|\bar{u}^{n+1}\|_2^2 - \|\bar{u}^n\|_2^2 + \|\bar{u}^{n+1} - \bar{u}^n\|_2^2 + 2\tau(\bar{u}^{n+1})^T A_1 \bar{u}^{n+1} &= 2\tau(\bar{u}^{n+1})^T \bar{1} \\ &\leq \tau \epsilon \lambda_1 \|\bar{u}^{n+1}\|_2^2 + \frac{\tau}{\epsilon \lambda_1} \|\bar{1}\|_2^2, \end{aligned}$$

where we have used Young inequality.

From point 1., $\epsilon(\bar{u}^{n+1})^T A_1 \bar{u}^{n+1} = \epsilon(\bar{u}^{n+1})^T A_1 \bar{u}^{n+1}$. Using $\lambda_1 \|\bar{u}^{n+1}\|_2^2 \leq (\bar{u}^{n+1})^T A_1 \bar{u}^{n+1}$ and $\|\bar{u}^{n+1} - \bar{u}^n\|_2^2 \geq 0$, we thus obtain

$$\|\bar{u}^{n+1}\|_2^2 - \|\bar{u}^n\|_2^2 + \tau \epsilon (\bar{u}^{n+1})^T A_1 \bar{u}^{n+1} \leq \tau \frac{\|\bar{1}\|_2^2}{\lambda_1 \epsilon}.$$

Summing from $n = 0$ to $M - 1$, we get

$$\|\bar{u}^M\|_2^2 - \|\bar{u}^0\|_2^2 + \tau \epsilon \sum_{n=0}^{M-1} (\bar{u}^{n+1})^T A_1 \bar{u}^{n+1} \leq M \tau \frac{\|\bar{1}\|_2^2}{\lambda_1 \epsilon}.$$

Since $u^0 = 0$ and $M\tau = T$, we get the desired inequality.

Question 4

The missing lines in the matlab file are :

- line 9 : `mat=speye(L*L,L*L)+tau*A;`
- Line 13 : `rhs=u+tau*b;`
- Line 15 : `u=gmpres(mat,rhs,1000,1.e-6,1000,lower,upper,u);`

To analyze the convergence of the method you should know the analytical solution of the problem you are facing. Just for the scope of clarity, in this case, since we are using a Backward FD for the time derivative and a Centered FD schema for the space derivative, we should expect a first order convergence in time and second order convergence in space.

In many situation, the analytical solution is not known. To see if the method has been arrived to convergence you can observe different quantities, one of those could be the L^∞ norm of the n solution.

You have to run many simulations to see when the norm of $\|u(\cdot, \cdot, 10)\|_{L^\infty(]0,1[^2)}$ is changing no more. In this exercise we fix the time step $M = 10$ and we run the simulation for different values of $L = 10, 20, 40, 80$.

For this simple problem you don't see any problem on making many simulations. But think about the case in which you have a more complex problem, how many resources and time this type of study may require.

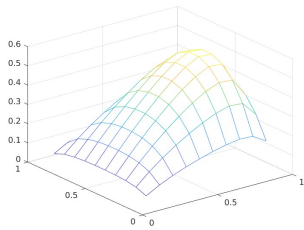


FIGURE 1 – Simulation run with $L = 10$ and $\|u(\cdot, \cdot, 10)\|_{L^\infty(]0,1[^2)} = 0.545$.

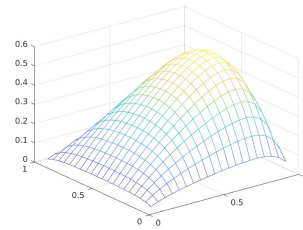


FIGURE 2 – Simulation run with $L = 20$ and $\|u(\cdot, \cdot, 10)\|_{L^\infty(]0,1[^2)} = 0.548$.

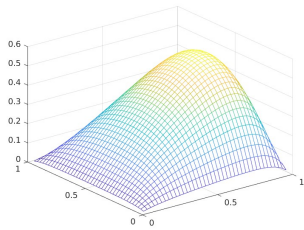


FIGURE 3 – Simulation run with $L = 40$ and $\|u(\cdot, \cdot, 10)\|_{L^\infty(]0,1[^2)} = 0.547$.

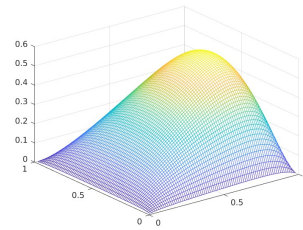


FIGURE 4 – Simulation run with $L = 80$ and $\|u(\cdot, \cdot, 10)\|_{L^\infty(]0,1[^2)} = 0.547$.