Homework #9

Combinatorial Number Theory (2025)

This homework is to be submitted on Moodle before next Tuesday at 23:59

E1. Show that for any $\epsilon > 0$ there exists a set $A \subseteq \mathbb{N}$ with $d(A) > 1 - \epsilon$ and such that A does not contain a subset of the form $t + FS((x_j)_{j=1}^{\infty})$ where $t \in \mathbb{N}$ and $(x_j)_{j \in \mathbb{N}}$ is a sequence of distinct elements in $\mathbb{N}$.
Hint: Set $A = \bigcap_{n \geq N} A_n$ for $N \in \mathbb{N}$ carefully chosen, where $A_n = \mathbb{N} \setminus (2^n\mathbb{N} + n)$.

> ***Solution:*** Consider $N \in \mathbb{N}$ such that $\frac{1}{2^{N-1}} < \epsilon$. Define $A_n \subseteq \mathbb{N}$ such that
>
> $$A_n^c = 2^n\mathbb{N} + n,$$
>
> with this, define
>
> $$A = \bigcap_{n \geq N} A_n.$$
>
> Notice that for every $n > N$, A_n cannot contain a set of the form $n + FS(x_j)_{j=1}^{\infty}$ for $n \geq N$. In fact, otherwise we would have that
>
> $$FS(x_j)_{j=1}^{\infty} \subseteq \mathbb{N} \setminus 2^n\mathbb{N},$$
>
> but this is not possible because of the pigeonhole principle (there is $i \in \{0, ..., 2^n - 1\}$ such that infinite many x_j have residue i module 2^n, therefore adding up 2^n of this terms we should get an element of $2^n\mathbb{N}$ which is a contradiction). Hence, A cannot contain a set of the form $t + FS(x_j)_{j=1}^{\infty}$ for any $t \in \mathbb{N}$ (if $t < N$ we can consider $t + FS(x_j)_{j=1}^{\infty} \supset t + \sum_{l=1}^{L} x_l + FS(x_j)_{j=l+1}^{\infty}$ for L large enough).
> Now observe that
>
> $$A = \bigcap_{n \geq N} \mathbb{N} \setminus (2^n\mathbb{N} + n) = \bigcap_{n=N}^{2^N+N-1} \mathbb{N} \setminus (2^n\mathbb{N} + n),$$
>
> (because for $n = 2^N + N$, $2^n\mathbb{N} + n \subseteq 2^N\mathbb{N} + N$). Now, notice that for $m < n \in \{N, ..., 2^N + N - 1\}$ we have that $2^n\mathbb{N} + n \cap 2^m\mathbb{N} + m = \emptyset$. Indeed, notice that if there exists $k, l \in \mathbb{N}$ with
>
> $$2^n k + n = 2^m l + m$$
>
> then we will have that
>
> $$2^m(l - 2^{n-m}k) = n - m,$$
>
> where the left side is greater than 2^m and the right side is less that 2^N, which is a contradiction. Thus
>
> $$d(A^c) = \sum_{n=N}^{2^N+N-1} d(2^n\mathbb{N} + n) = \sum_{n=N}^{2^N+N-1} \frac{1}{2^n} = \frac{1}{2^{N-1}}(1 - \frac{1}{2^{2^N}}) \leq \frac{1}{2^{N-1}} < \epsilon.$$

Therefore, $d(A) \geq 1 - \epsilon$.

Criteria(Mathematical correctness):

- Proved that A doesn't contain a set of the form $t + FS(x_j)_{j=1}^{\infty}$ — 2 points
- Shown that $d(A^c) < \varepsilon$ (bound + the exact formula or a proof of the existence) — 2 points
- Suitable choice of N — 1 point