Homework #8
Combinatorial Number Theory (2025)

This homework is to be submitted on Moodle before next Tuesday at 23:59

E1. Prove the direction of Rado's theorem not seen in the lecture: Let A be a $q \times k$ integer matrix not satisfying the columns condition. Show that the equation

$$A \cdot x = 0$$

is not partition regular.

Solution: We will show that if the system $A \cdot x = 0$ is partition regular, then A satisfies the column condition.

Similarly to the proof for a single equation of Rado's theorem, we can find a subset $J_0 \subseteq [k]$ such that

$$\sum_{j \in J_0} \mathbf{a}_j = 0.$$

[[We repead the argument here for the sake of completeness: Let p be any prime bigger than $\sum_{i,j} |a_{i,j}|$. Note that every positive integer $n \in \mathbb{N}$ can be uniquely written as $n = p^s m$ for some $s \geqslant 0$ and some $m \in \mathbb{N}$ with $\gcd(p, m) = 1$. Define a coloring $\chi : \mathbb{N} \to \{1, 2, \ldots, p-1\}$ via

$$\chi(n) = m \bmod p$$

where m is the unique number in the decomposition $n = p^s m$ described above. We claim that this coloring does not admit a monochromatic solution to the equation $Ax = 0$ unless $\sum_{j \in J} \mathbf{a}_j = 0$ for some non-empty set $J \subseteq \{1, \ldots, k\}$. Indeed, suppose $x_1, \ldots, x_k \in \mathbb{N}$ with $\chi(x_1) = \ldots = \chi(x_k)$ and $\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \ldots + \mathbf{a}_k x_k = 0$. We can write each x_j uniquely as $x_j = p^{s_j} m_j$ with $\gcd(p, m_j) = 1$. Consider $s = \min\{s_1, \ldots, s_k\}$ and define $J = \{1 \leqslant j \leqslant r : s_j = s\}$. Taking $y_j = x_j p^{-s}$, we note that $\chi(x_j) = \chi(y_j)$ and hence $\chi(y_1) = \ldots = \chi(y_k)$ and $\mathbf{a}_1 y_1 + \mathbf{a}_2 y_2 + \ldots + \mathbf{a}_k y_k = 0$. We thus have

$$0 = \mathbf{a}_1 y_1 + \mathbf{a}_2 y_2 + \ldots + \mathbf{a}_k y_k \equiv \sum_{j \in J} \mathbf{a}_j m_j \bmod (\mathbb{Z}/p\mathbb{Z})^q,$$

and hence $\sum_{j \in J} \mathbf{a}_j \equiv 0 \bmod (\mathbb{Z}/p\mathbb{Z})^q$ because all m_j take on the same non-zero residue mod p. Since $p > \sum_{i,j} |a_{i,j}|$, from $\sum_{j \in J} \mathbf{a}_j \equiv 0 \bmod \mathbb{Z}_p^q$ we deduce that $\sum_{j \in J} \mathbf{a}_j = 0$ as claimed.]]

If $J_0 = [k]$, then we have finished. Assume that we have found nonempty pairwise disjoint sets $J_0, \ldots, J_\ell \subseteq [k]$ such that for each $i \in \{1, \ldots, l\}$ and $i \in J_i$, there are rational numbers $\lambda_j^{(i)}$, such that

$$\sum_{j \in J_0} \mathbf{a}_j = 0,$$

$$\sum_{j \in J_1} \mathbf{a}_j = \sum_{j \in J_0} \lambda_j^{(1)} \mathbf{a}_j,$$

$$\vdots$$

$$\sum_{j \in J_\ell} \mathbf{a}_j = \sum_{j \in J_0 \cup J_1 \cup \ldots \cup J_{\ell-1}} \lambda_j^{(\ell)} \mathbf{a}_j,$$

and $J_0 \cup \cdots \cup J_\ell \neq [k]$. Let $V = \mathrm{span}_{\mathbb{Q}}(\{\mathbf{a}_j : j \in J_0 \cup \cdots \cup J_\ell\})$ and define the natural projection $\pi : \mathbb{Q}^q \to \mathbb{Q}^q/V \cong \mathbb{Q}^m$. Consider B the matrix with columns in $\mathbb{Q}^m$ given by $\pi(\mathbf{a}_j)$ for $j \in [k] \setminus (J_0 \cup \cdots \cup J_\ell)$. We observe that a solution of the system $A \cdot x = 0$ in $\mathbb{N}^k$ induces a solution of the system $B \cdot y = 0$ in $\mathbb{N}^{k-|J_0 \cup \cdots \cup J_\ell|}$. Thus, the system $B \cdot y$ is partition regular (as restricting a monochromatic x to the remaining indices gives a monochromatic y). We can assume without loss of generality that the coordinates of B are integer (by replacing B by nB for $n \in \mathbb{N}$ big enough).

Repeating the same argument as before, we get that there exists $J_{\ell+1} \subseteq [k] \setminus J_0 \cup \cdots \cup J_\ell$ such that

$$\sum_{j \in J_{\ell+1}} \mathbf{b}_j = 0 \bmod V.$$

In other words, we have that

$$\sum_{j \in J_{\ell+1}} \mathbf{a}_j \in \mathrm{span}_{\mathbb{Q}}(\{\mathbf{a}_j : j \in J_0 \cup \cdots \cup J_\ell\}),$$

which concludes the induction.

Thus, by finiteness of $[k]$, this induction finishes in finitely many steps, giving a partition $J_0, \ldots, J_\ell$ of $[k]$ satisfying the column condition.

Note: It is also possible to use a expansion of digits modulo p to extent the one-equation argument to the general case.