

Homework #7

Combinatorial Number Theory (2025)

This homework is to be submitted on Moodle before next Tuesday at 23:59

- E1.** Show that for every $k \in \mathbb{N}$ and $m \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that for each prime number $p \geq N$ there exists a sequence of m consecutive numbers $n, n + 1, \dots, n + m - 1 \in \{1, 2, \dots, p - 1\}$ all of which are k -th power residues mod p .

Solution: For a prime p with $d = \gcd(k, p - 1)$, the k -th power residues modulo p form a subgroup H of $(\mathbb{Z}/p\mathbb{Z})^*$ with $|H| = (p - 1)/d$. The group partitions into $d \leq k$ cosets of H .

By Brauer's theorem, if p is sufficiently large, then we can find a and r such that $\{a, a + r, \dots, a + (m - 1)r\} \cup r$ are all in the same coset. Multiplying by a common difference r we obtain that

$$\{ar^{-1}, ar^{-1} + 1, \dots, ar^{-1} + m - 1\}$$

are m consecutive numbers that are all in the same coset as $1 = rr^{-1}$, i.e. are all k -th power residues.

Criteria(Mathematical correctness):

- Noticed that k -th power residues mod p form a subgroup in $(\mathbb{Z}/p\mathbb{Z})^*$ — 1 point
- Colored \mathbb{N} according to a cosets — 1 point
- Proved that number of cosets is bounded by k — 1 point
- Applied Brauer's theorem — 1 point
- Divided AP by d and concluded — 1 point