# Homework #6

**Combinatorial Number Theory (2025)**

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*This homework is to be submitted on Moodle before next Tuesday at 23:59*

**E1.** Let $p \in \beta\mathbb{N}$ be an idempotent ultrafilter.

**(a)** Show that $p$ cannot be a principal ultrafilter.

> ***Solution:*** If $p$ were principal, namely $p = \delta_n$, then $\delta_n = \delta_n + \delta_n = \delta_{2n}$. Therefore $2n = n$ which implies $n = 0$, a contradiction.

**(b)** Show that for any $a \in \mathbb{N}$, $a\mathbb{N} \in p$.

> ***Solution:*** Since $\mathbb{N} \in p$, by partition regularity of ultrafilters, there is a unique $i \in \{1, ..., a-1\}$ such that $a\mathbb{N} + i \in p$. As $p = p + p$, we have that
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> $$a\mathbb{N} = \{m \in \mathbb{N} \mid a\mathbb{N} + i - m \in p\} \in p,$$
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> which gives the result.

**E2.** Let $T : X \to X$ be a continuous transformation, where $(X, d)$ is a compact metric space. Assume that $T$ has the following property: For $x, y \in X$,

$$\inf\{d(T^n(x), T^n(y)) : n \geq 1\} = 0 \Longrightarrow x = y. \tag{1}$$

Prove that $T$ is bijective.
**Hint**: For an idempotent $p \in \beta\mathbb{N}$ and a point $x \in X$, prove that $p\text{-}\lim_{n\in\mathbb{N}} T^n x = x$.

> ***Solution:*** The map $T$ is clearly injective, since distality implies that for $x, y \in X$ with $T(x) = T(y)$, we have $x = y$. To prove surjectivity, we follow the hint. Call $y = p\text{-}\lim_{n\in\mathbb{N}} T^n x$. By Exercise 2 from Exercise Sheet 6, we have that
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> $$p\text{-}\lim_{n\in\mathbb{N}} T^n y = p\text{-}\lim_{n\in\mathbb{N}} T^n p\text{-}\lim_{m\in\mathbb{N}} T^m x = p\text{-}\lim_{n\in\mathbb{N}} p\text{-}\lim_{n\in\mathbb{N}} T^{n+m} x = p\text{-}\lim_{n\in\mathbb{N}} T^n x = y.$$
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> Thus $p\text{-}\lim_{n\in\mathbb{N}} T^n x = y = p\text{-}\lim_{n\in\mathbb{N}} T^n y$ and by distality that implies that $x = y$. In particular $x = T(p\text{-}\lim_{n\in\mathbb{N}} T^{n-1} x)$, concluding that $T$ is surjective. Therefore, $T$ is bijective.

***Criteria(Mathematical correctness):***

> - Injectivity — 1 point
> - Shown that $y = p\text{-}\lim_{n\in\mathbb{N}} T^n y$ — 1 point
> - Shown that $x = y$ — 2 points
> - Concluded the surjectivity of $T$ using that $T$ is continuous — 1 point