Homework #4
Combinatorial Number Theory (2025)

This homework is to be submitted on Moodle before next Tuesday at 23:59

E1. Use the finitary form of van der Waerden's theorem to show the following: if $E \subseteq \mathbb{N}^d$ is piecewise syndetic[1], then for any $k \in \mathbb{N}$ and any $\mathbf{v} \in \mathbb{N}^d$, there exists $n \in \mathbb{N}$ such that

$$E \cap (E - n\mathbf{v}) \cap \cdots \cap (E - kn\mathbf{v})$$

is piecewise syndetic.

Solution: Write $E = T \cap S$ with T thick and S syndetic. Let $F = \{f_1, \ldots, f_r\} \subseteq \mathbb{N}^d$ such that $S - F \supseteq \mathbb{N}^d$. Set $l = W(r, k+1)$. Since T is thick, it is easy to show that $T - f_1 \cap \cdots \cap T - f_r$ is also thick. This implies that the set

$$\mathcal{T} := \{x \in \mathbb{N}^d : \{x + i\mathbf{v} : i = 1, \ldots, l\} \subseteq T - f_1 \cap \cdots \cap T - f_r\}$$

is thick as well, in particular piecewise syndetic.

Take $x \in \mathcal{T}$. Define the coloring $\chi : \{1, \ldots, l\} \to r$ such that $\chi(i) = j$ if and only if j is the minimum such that $x + i\mathbf{v} \in S - f_j$. By finitistic van der Waerden's theorem, there is a monochromatic k-arithmetic progression in $\{a, a+b, \ldots, a+kb\}$ in $[l]$, where $b = b(x) \in [l]$ is the minimal with this property. This implies that there is $i \in [r]$ such that

$$f_i + x + (a + jb)\mathbf{v} \subseteq S, \ \forall j \in \{0, \ldots, k\}.$$

Thus, we conclude that

$$f_i + x + (a + jb)\mathbf{v} \subseteq E \ \forall j \in \{0, \ldots, k\},$$

which implies that

$$x \in \bigcup_{(i,m) \in [r] \times [l]} (E \cap (E - b\mathbf{v}) \cap \cdots \cap (E - kb\mathbf{v})) - m\mathbf{v} - f_i.$$

Now, for each $n \in [l]$ define $\mathcal{T}(n) = \{x \in \mathcal{T} : b(x) = n\}$. Notice that for each $n \in [l]$

$$\mathcal{T}(n) \subseteq \bigcup_{(i,m) \in [r] \times [l]} (E \cap (E - n\mathbf{v}) \cap \cdots \cap (E - kn\mathbf{v})) - m\mathbf{v} - f_i. \quad (1)$$

As

$$\mathcal{T} = \bigcup_{n \in [l]} \mathcal{T}(n),$$

and since piecewise syndetic sets are partition regular (same proof as for $d = 1$ in the lecture notes), we have that there is $n \in [l]$ such that $\mathcal{T}(n)$ is piecewise syndetic. This and equation

[1] A set $T \subseteq \mathbb{N}^d$ is *thick* if for every finite set $F \subseteq \mathbb{N}^d$, there is $n \in \mathbb{N}^d$ such that $n + F \subseteq T$. A set $S \subseteq \mathbb{N}^d$ is *syndetic* if there is a finite set $F \subseteq \mathbb{Z}^d$ such that $\mathbb{N}^d \subseteq F + S$. A set $P \subseteq \mathbb{N}^d$ is *piecewise syndetic* if $P = T \cap S$ with T thick and S syndetic.

(1) imply that $E \cap (E - n\mathbf{v}) \cap \cdots \cap (E - kn\mathbf{v})$ is piecewise syndetic, finishing.
Note: Finitely many translates of $\mathcal{T}(n)$ yield to a thick set, which implies that finitely many translates of $E \cap (E - n\mathbf{v}) \cap \cdots \cap (E - kn\mathbf{v})$ yield to a thick set as well. It is a good exercise to prove that if F is a finite set containing 0, and $T := F + A$ is thick, then $A = T \cap S$ where $S = A \cup T^c$ is syndetic.

E2. Suppose $\mathbb{N} = \bigcup_{i=1}^r C_i$. Prove that for some $i \in \{1, \ldots, r\}$, the set C_i is both AP-rich and GP-rich.
Hint: Use Gallai's theorem in 2 dimensions.

Solution: Let $c : \mathbb{N} \to \{1, \ldots, r\}$ be the coloring of $\mathbb{N}$ (with r colors) defined by $c(n) = \min\{1 \leq i \leq r : n \in C_i\}$. We define a coloring $\chi : \mathbb{N}^2 \to \{1, \ldots, r\}$ by $\chi(n, m) = c(n \cdot 2^m)$. For $k, l \in \mathbb{N}$, let $V_{k,l} = \{(i, j) : 0 \leq i \leq k-1, 0 \leq j \leq l-1\}$. By Gallai's theorem, there is a point $(a, b) \in \mathbb{N}^2$ and a dilation $d \in \mathbb{N}$ such that $(a, b) + d \cdot V_{k,l}$ is monochromatic with respect to χ. From the definition of χ, this means that the set

$$\left\{(a + id)2^{b+jd} : 0 \leq i \leq k-1, 0 \leq j \leq l-1\right\}$$

is monochromatic with respect to c. Notice that for fixed j,

$$\left\{(a + id)2^{b+jd} : 0 \leq i \leq k-1\right\} = \left\{2^{b+dj}a + i(2^{b+jd}d) : 0 \leq i \leq k-1\right\}$$

is an arithmetic progression of length k. Similarly, for fixed i,

$$\left\{(a + id)2^{b+jd} : 0 \leq j \leq l-1\right\} = \left\{2^b(a + id) \cdot \left(2^d\right)^j : 0 \leq j \leq l-1\right\}$$

is a geometric progression of length l. Since k and l were arbitrary, this completes the proof.

Criteria(Mathematical correctness):

- If a proof works for $\mathbb{N}^2$, the student receives the full score, despite the possible errors in the $\mathbb{Z}^2$ case.