Homework #3
Combinatorial Number Theory (2025)

This homework is to be submitted on Moodle before next Tuesday at 23:59

E1. Prove that for any finite coloring of $\mathbb{N}$ and any $k \in \mathbb{N}$, there is a monochromatic k-cube

$$x_0 + \left\{ \sum_{i=1}^{k} \epsilon_i x_i : \epsilon_1, \ldots, \epsilon_k \in \{0,1\} \right\}$$

for some $x_0, x_1, \ldots, x_k \in \mathbb{N}$.

> ***Solution:*** Let $\chi : \mathbb{N} \to \{1, \ldots, r\}$ be a finite coloring of $\mathbb{N}$. Since piecewise syndetic sets are partition regular, there is a color $i \in \{1, \ldots, r\}$ such that $A = \{n \in \mathbb{N} : \chi(n) = i\}$ is piecewise syndetic.
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> By Corollary 30 from the lecture notes, let $x_1 \in \mathbb{N}$ such that $A_1 = A \cap (A - x_1)$ is piecewise syndetic. For each $n \in \mathbb{N}$, applying Corollary 30 again, choose $x_n \in \mathbb{N}$ such that $A_n = A_{n-1} \cap (A_{n-1} - x_n)$ is piecewise syndetic.
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> Now let $k \in \mathbb{N}$. Choosing
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> $$x_0 \in A_k = \bigcap_{\epsilon_1, \ldots, \epsilon_k \in \{0,1\}} \left(A - \sum_{i=1}^{k} \epsilon_i x_i \right)$$
>
> produces a k-cube $x_0 + \left\{ \sum_{i=1}^{k} \epsilon_i x_i : \epsilon_1, \ldots, \epsilon_k \in \{0,1\} \right\}$ in the monochromatic set A.

E2. Let

$$\mathcal{P} = \{A \subset \mathbb{N} : \text{for each infinite set } B \subseteq \mathbb{N},\ \exists b_1 < b_2 < b_3 \in B \text{ such that } b_1 + b_2^2 + b_3^3 \in A\}.$$

Prove that $\mathcal{P}$ is closed under finite intersections.

> ***Solution:*** First, we notice that $\mathcal{P}$ is upward closed. We have proved in the lecture that a upward closed family is closed under finite intersections if and only if its dual family is partition regular. So instead of proving that $\mathcal{P}$ is closed under finite intersections, it suffices to show that $\mathcal{P}^*$ is partition regular. Using the definition of the dual family, it is easy to see that
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> $$\mathcal{P}^* = \{A \subset \mathbb{N} : \text{there is an infinite set } B \subseteq \mathbb{N},\ \forall b_1 < b_2 < b_3 \in B \text{ such that } b_1 + b_2^2 + b_3^3 \in A\}.$$
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> To prove that $\mathcal{P}^*$ is partition regular, we must show that for any set $A \in \mathcal{P}^*$ and for any partitioning of A into two sets, i.e., for any disjoint sets D, C with $A = D \cup C$, either D belongs to $\mathcal{P}^*$ or C belongs to $\mathcal{P}^*$. Since $A \in \mathcal{P}^*$, there exist an infinite set $B \subseteq \mathbb{N}$ with $\{b_{i_1} + b_{i_2}^2 + b_{i_3}^3 : i_1 < i_2 < i_3\} \subset A$. Define a 2-coloring χ of all 3-subsets of $\mathbb{N}$ via
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> $$\chi(\{i_1, i_2, i_3\}) = \begin{cases} \textit{blue}, & \text{if } b_{i_1} + b_{i_2}^2 + b_{i_3}^3 \in D \\ \textit{red}, & \text{if } b_{i_1} + b_{i_2}^2 + b_{i_3}^3 \in C. \end{cases}$$

Since $b_{i_1} + b_{i_2}^2 + b_{i_3}^3$ always belongs to A, and A is the disjoint union of D and C, this coloring is well defined. By Ramsey's Theorem for 2-sets, there exists an infinite sequence $i_1 < i_2 < \ldots \in \mathbb{N}$ such that $\chi(\{i_k, i_\ell, i_j\})$ always has the same color for all $k < \ell < j \in \mathbb{N}$. If this color is blue then, by the definition of χ, this means that $\{b_{i_k} + b_{i_\ell}^2 + b_{i_j}^3 : k < \ell < j\}$ is a subset of D. Conversely, if the color is red then we have that $\{b_{i_k} + b_{i_\ell}^2 + b_{i_j}^3 : k < \ell < j\}$ is a subset of C. This shows that either D or C belongs to $\mathcal{P}^*$, completing the proof.

Criteria(Mathematical correctness):

- Shown that it's suffices to prove that $\mathcal{P}^*$ is partition regular — 1 point.
- The dual family $\mathcal{P}^*$ is found correctly — 1 point.
- Defined a right coloring on B — 1 point.
- Used Ramsey's theorem — 1 point.
- Completed the proof — 1 point.