Homework #2

Combinatorial Number Theory (2025)

This homework is to be submitted on Moodle before next Tuesday at 23:59

E1. Let $K \subset \mathbb{R}^3$ be infinite. Show that K contains either an infinite coplanar subset or an infinite subset of points in convex position whose convex hull is non-degenerate[1].

> ***Solution:*** Assume that there is no plane that contains infinite points from K. Let $\mathcal{P}$ be any plane in $\mathbb{R}^3$. Let $K_{\mathcal{P}}$ be the orthogonal projection of K on $\mathcal{P}$. Notice that $K_{\mathcal{P}}$ has to be infinite, since two points which share the same projection are contained in the same line. Thus, as there are no infinite coplanar points, $K_{\mathcal{P}}$ is infinite. By Erdős-Szekeres theorem, there are either infinite points in $K_{\mathcal{P}}$ which are collinear, or there are infinite points in convex position. The first option is impossible, since that would imply that infinite points in K are contained in the plane orthogonal to such line. Thus, there is an infinite set $B_{\mathcal{P}} \subseteq K_{\mathcal{P}}$ of points in convex position. For each $b_{\mathcal{P}} \in K_{\mathcal{P}}$ pick a one $b \in K$ such that the projection of b to $\mathcal{P}$ is $b_{\mathcal{P}}$. Call $B \subseteq K$ the set of such b's. Clearly the points in B are in convex position, as their projections $B_{\mathcal{P}}$ to the plane $\mathcal{P}$ are. Since the points in B are not contained in a plane by hypothesis, their convex hull must be non-degenerate, finishing.

E2. Prove that for every $k \geq 3$, there exists some N such that among any N points in the plane, there are either k points lying on a line, or k points in convex position.

> ***Solution:*** This problems boils down to repeating the proof of Erdős-Szekeres' Theorem using solely Ramsey's theorem in hypergraph (i.e. the finitary version of Ramsey's theorem in k-sets). Let $R = R_3(k, 2)$ and $N = R_3(R, 2)$. We $[N]^{(3)}$ by assigning the color red to $\{\vec{x}, \vec{y}, \vec{z}\} \in [N]^{(3)}$ if the points are collinear, and the color blue otherwise. By Ramsey's theorem for 3-sets (finitary version) there exists a set $L \subseteq K$ with cardinal R such that all 3-sets in L have the same color. If this color is red, then any three distinct points in L are collinear, which can only happen if all the points in L lie on a straight line, in which case we are done as $R \geq k$.
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> On the other hand, if all elements in $L^{(3)}$ are blue, then we can define a new coloring. We assing the color red to $\{\vec{x}, \vec{y}, \vec{z}\} \in L^{(3)}$ if the triangle form by such points contains an even number of points from L, and color blue otherwise. Again, by Ramsey theorem for 3-sets, there is a subset $C \subseteq L$ with $|C| \geq k$ such that all three points are of the same color. The rest of the proof follows exactly as in the lecture notes.

Criteria(Mathematical correctness):

> - Defined the right colorings — 1 point for each coloring
> - Reduced the second case to showing that 4 points are in convex position (by Carathéodory's

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[1] Non-degenerate means that is not contained in an hyperplane. Equivalently, that the convex hull of the points has positive volume

theorem) — 1 point

- Completed the proof — 2 points