Homework #14
Combinatorial Number Theory (2025)

This homework is to be submitted on Moodle before next Tuesday at 23:59

For $A \subseteq \mathbb{Z}$ we define

$$d^*(A) = \lim_{n\to\infty} \left(\max_{k\in\mathbb{Z}} \frac{|A \cap \{k+1, \ldots, k+n\}|}{n} \right)$$

E1. Show the following strengthening of Jin's theorem: Let $A, B \subseteq \mathbb{Z}$ with $d^*(A) = \alpha$ and $d^*(B) = \beta$. Then, there is a finite set F of cardinality $|F| \leq 1/(\alpha\beta)$ such that $(A - B) + F$ is thick.
Hint: Find sequences of elements $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, and $(z_n)_{n\in\mathbb{N}}$ such that

- if $A_n = A \cap [x_n + 1, x_n + n^2]$ then $\lim_{n\to\infty} |A_n|/n^2$,
- if $B_n = B \cap [y_n + 1, y_n + n]$, then $\lim_{n\to\infty} |B_n|/n = \beta$,
- if $G_n = A_n - x_n \subseteq [n^2]$ and $H_n = B_n - y_n \subseteq [n]$, then $E_n = (G_n - z_n) \cap H_n \subseteq [n]$ is such that $\lim_{n\to\infty} |E_n|/n \geq \alpha\beta$.

Hint 2: It may be useful to use Exercises 2 and 3 from Exercise sheet 14.

Solution: By the definition of upper Banach density, we can pick two sequences of integers $\{x_n \mid n \in \mathbb{N}\}$ and $\{y_n \mid n \in \mathbb{N}\}$ such that, if we let:

$$A_n = A \cap \left[x_n + 1, x_n + n^2\right]$$
$$B_n = B \cap [y_n + 1, y_n + n]$$

then $\lim_{n\to\infty} |A_n| /n^2 = \alpha$ and $\lim_{n\to\infty} |B_n| /n = \beta$. For every n we shall find a suitable shift of A_n that meets B_n on a set whose relative density approaches $\alpha\beta$ as n goes to infinity.

For every n, apply exercise 2 from TA 14 where $G = A_n - x_n \subseteq \left[1, n^2\right]$ and $H = B_n - y_n \subseteq [1, n]$. Clearly, $|G| = |A_n|$ and $|H| = |B_n|$. Then we can pick a suitable sequence $\{z_n \mid n \in \mathbb{N}\}$ such that

$$\frac{|(A_n - x_n - z_n) \cap (B_n - y_n)|}{n} \geqslant \frac{|A_n|}{n^2} \cdot \frac{|B_n|}{n} - \frac{|B_n|}{n^2}.$$

Now let:

$$E_n = (A_n - x_n - z_n) \cap (B_n - y_n) \subseteq [1, n].$$

By passing the previous inequality to the limit, we obtain $\lim_{n\to\infty} |E_n| /n \geqslant \alpha\beta$.

By Exercise 3 from TA 14 where $\gamma = \alpha\beta > 0$, we can pick a finite set F of cardinality $|F| \leqslant 1/\alpha\beta$ and such that for every m there exists n (in fact, infinitely many n) such that:

$$[1, m] \subseteq (E_n - E_n) + F \subseteq (A_n - x_n - z_n) - (B_n - y_n) + F \subseteq (A - B) + F - t_n,$$

and hence $[t_n + 1, t_n + m] \subseteq (A-B)+F$, where we denoted $t_n = x_n - y_n + z_n$. This completes the proof that $(A - B) + F$ is thick.