Homework #13
Combinatorial Number Theory (2025)

This homework is to be submitted on Moodle before next Tuesday at 23:59

For $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ the integer part of x, this is, the largest integer $k \in \mathbb{Z}$ such that $k \leq x$.

E1. Prove that $R := \{\lfloor \lfloor \sqrt{2}n \rfloor \sqrt{2} \rfloor\}_{n \in \mathbb{N}}$ is not an intersective set.
Hint: Find a explicit expression of $r_n = \lfloor \lfloor \sqrt{2}n \rfloor \sqrt{2} \rfloor$ when $\{\sqrt{2}n\} > 1/\sqrt{2}$ and $\{\sqrt{2}n\} \leq 1/\sqrt{2}$.

> ***Solution:*** We denote $\{x\} := x - \lfloor x \rfloor$.
> We consider the sequence $r_n = \lfloor \sqrt{2} \lfloor \sqrt{2}n \rfloor \rfloor$. Notice that $r_n = \lfloor 2n - \sqrt{2}\{\sqrt{2}n\} \rfloor$, and thus
>
> $$r_n = \begin{cases} 2n - 1 & \text{if } \{\sqrt{2}n\} \leq 1/\sqrt{2} \\ 2n - 2 & \text{if } \{\sqrt{2}n\} > 1/\sqrt{2} \end{cases}. \tag{1}$$
>
> If by contradiction $R = \{r_n\}_n$ is intersective, then by partition regularity (exercise 3 from Exercise list 13) $R \cap 2\mathbb{Z}$ is intersective since $2\mathbb{Z} + 1$ is not intersective. Thus, by exercise 1 from the Exercise list 12, there exists $r_n \in R \cap 2\mathbb{Z}$
>
> $$\|r_n \frac{\sqrt{2}}{2}\| \leq \varepsilon$$
>
> for some $\varepsilon > 0$ to be determined. Since $r_n \in 2\mathbb{Z}$, this can be written as
>
> $$\{\sqrt{2}n\} > 1/\sqrt{2}, \text{ and } ||(2n-2)\frac{\sqrt{2}}{2}|| \leq \varepsilon.$$
>
> This implies that:
>
> $$\begin{aligned} \varepsilon &\geq ||(2n-2)\frac{\sqrt{2}}{2}|| \\ &= ||\sqrt{2}n - (\sqrt{2} - 1)|| \\ &= \{\sqrt{2}n\} - (\sqrt{2} - 1), \end{aligned}$$
>
> where we used that $\{\sqrt{2}n\} > 1/\sqrt{2} > (\sqrt{2} - 1)$. By taking $\varepsilon = 1/\sqrt{2} - (\sqrt{2} - 1)$, this leads to contradiction.

> ***Criteria(Mathematical correctness):***
>
> - Found an explicit expression of r_n — 1 point.
> - Proved that $R \cap (2\mathbb{Z} + 1)$ is not intersective — 1 point.
> - Proved that $R \cap 2\mathbb{Z}$ is not intersective — 2 points.
> - Concluded by partition regularity — 1 point.