Homework #12
Combinatorial Number Theory (2025)

This homework is to be submitted on Moodle before next Tuesday at 23:59

E1. Let $(a_n)_{n\in\mathbb{N}}$ a sequence in $\mathbb{N}$ such that

$$\inf\left\{\frac{a_{n+1}}{a_n} : n \in \mathbb{N}\right\} > 1.$$

Show that the set $A = \{a_n : n \in \mathbb{N}\}$ is not intersective.

Solution: Call $\lambda := \inf\left\{\frac{a_{n+1}}{a_n} : n \in \mathbb{N}\right\}$. First, assume $\lambda > 10$. Let

$$X_n = \left\{\alpha \in [0,1] : \|a_n\alpha\| \geq \frac{1}{10}\right\} = \bigcup_{j=0}^{a_n-1}\left(\frac{j}{a_n} + \left[\frac{1}{10a_n}, \frac{9}{10a_n}\right]\right).$$

We claim $X = \bigcap_{n\in\mathbb{N}} X_n \neq \emptyset$. Let $I_1 = \left[\frac{1}{10a_1}, \frac{9}{10a_1}\right] \subseteq X_1$. Suppose we have chosen $0 \leq j_k \leq a_k - 1$ for $1 \leq k \leq n$ such that the intervals $I_k = \frac{j_k}{a_k} + \left[\frac{1}{10a_k}, \frac{9}{10a_k}\right]$ satisfy $I_1 \supseteq \cdots \supseteq I_n$. The interval I_n has length $|I_n| = \frac{8}{10a_n} > \frac{8}{a_{n+1}}$. Therefore, for some $j_{n+1} \in \{0, 1, \ldots, a_{n+1} - 1\}$, the points $\frac{j_{n+1}}{a_{n+1}}$ and $\frac{j_{n+1}+1}{a_{n+1}}$ belong to I_n. In particular,

$$I_{n+1} = \frac{j_{n+1}}{a_{n+1}} + \left[\frac{1}{10a_{n+1}}, \frac{9}{10a_{n+1}}\right] \subseteq I_n.$$

Since $I_1 \supseteq I_2 \supseteq \ldots$ is a nested sequence of non-empty compact sets, the intersection $\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$. Moreover, this intersection is a subset of X, so $X \neq \emptyset$.
Let $\alpha \in X$. Then $\|a_n\alpha\| \geq \frac{1}{10}$ for each $n \in \mathbb{N}$, so A is not intersective by Exercise 1 from Exercise sheet 12.

Note: the following will not be evaluated:

Now suppose $\lambda > 1$ is arbitrary. Let $k \in \mathbb{N}$ such that $\lambda^k > 10$. Partition

$$A = \bigcup_{i=0}^{k-1} A_i$$

with $A_i = \{a_i, a_{i+k}, a_{i+2k}, \ldots\}$. We have $a_{i+(n+1)k} \geq \lambda^k a_{i+nk} > 10a_{i+nk}$, so A_i is not intersective. By Exercise 1 from Exercise sheet 13, it follows that A is not intersective.

Criteria(Mathematical correctness):

- Proved that it suffices to find α such that $\|a_n\alpha\| \geq \frac{1}{10}$ for each $n \in \mathbb{N}$ — 1 point.
- Described sets X_k such that $\|a_k\alpha\| \geq \frac{1}{10}$ — 1 point.
- Defined a sequence of nested intervals (with a proof) — 2 points.
- Concluded by compactness — 1 point.