

# Homework #11

## Combinatorial Number Theory (2025)

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*This homework is to be submitted on Moodle before next Tuesday at 23:59*

**E1.** Let  $0 < \alpha < 1$ . Show that there is a constant  $C = C(\alpha) > 0$  such that for  $N$  sufficiently large if  $A \subset \{1, \dots, N\}$  is such that  $|A| \geq \alpha N$ , then

$$\frac{1}{N^2} \sum_{a,d=1}^N \mathbb{1}_A(a) \mathbb{1}_A(a+d) \mathbb{1}_A(a+2d) \geq C.$$

**Hint:** Let  $M$  be such that any subset of  $\{1, \dots, M\}$  of density at least  $\alpha/2$  contains a 3AP. Why such  $M$  exists? Then divide  $\{1, \dots, N\}$  into disjoint APs with appropriate common difference and length (depending on  $N$  and  $M$ ).

**Solution:** By Roth's theorem, there exists  $M = M(\alpha)$  such that any subset of  $\{1, \dots, M\}$  of density at least  $\alpha/2$  contains a 3AP. Suppose that  $N$  is sufficiently large compared to  $\alpha$ . We let  $d \leq N/10M$  and we divide  $\{1, \dots, N\}$  into disjoint progressions of common difference  $d$  and length between  $M$  and  $2M$ . Let  $P(d)$  denote the number of these progressions on which  $A$  has density at least  $\alpha/2$ . Then we have

$$|A| \leq 2MP(d) + \frac{\alpha}{2}N,$$

and so

$$P(d) \geq \frac{\alpha N}{4M}.$$

On each such progression, there is a 3AP which belongs in  $A$ , thus, we get

$$\sum_{d \leq N/10M} P(d) \geq \frac{N}{10M} \frac{\alpha N}{4M} = \frac{\alpha N^2}{40M^2}$$

3APs in total.

Note that we might have counted the same progression multiple times. Suppose that  $\{x, x+y, x+2y\}$  occurs more than once, i.e., it occurs on  $P(d)$  for more than one choices of  $d$ . If this progression is contained in a progression of common difference  $d$  and length between  $M$  and  $2M$ , then  $y \in \{d, 2d, \dots, 2Md\}$ , thus we have at most  $2M$  choices for  $d$ . This shows that each 3AP is counted at most  $2M$  times. Therefore, we get at least

$$\frac{1}{2M} \sum_{d \leq N/10M} \geq \frac{\alpha N^2}{80M^3}$$

distinct 3APs in  $A$ . Taking  $C = C(\alpha) = \alpha/80M^3$  concludes the proof.