Exercise Set #9

Combinatorial Number Theory (2025)

E1. Compute the densities of the following sets:

(i) $\{17 + 8n : n \in \mathbb{N}\}$.

(ii) $\{2n + (-1)^n n : n \in \mathbb{N}\}$.

(iii) The set of all natural numbers that are divisible by 5 but not divisible by 3.

(iv) $\{\lfloor n\alpha + \beta \rfloor : n \in \mathbb{N}\}$ for $\alpha, \beta \geq 0$.

Solution: (i) The density of $\{17+8n : n \in \mathbb{N}\}$ is $\frac{1}{8}$. To see that one can use Proposition 54 and conclude that

$$d\left(\{17 + 8n : n \in \mathbb{N}\}\right) = \lim_{n\to\infty} \frac{n}{17 + 8n} = \frac{1}{8}.$$

(ii) Notice that $2n + (-1)^n n$ is equal to n if n is odd and to $3n$ if n is even. Thus

$$\{2n + (-1)^n n : n \in \mathbb{N}\} = \{1, 3, 5, 7, \dots\} \cup \{6, 12, 18, 24, \dots\},$$

which is a disjoint union of sets of densities $\frac{1}{2}$ and $\frac{1}{6}$ respectively (by a similar reasoning than for the first set). Therefore the total density is $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$.

(iii) Since the density of $\{5, 10, 15, 20, \ldots\}$ is $\frac{1}{5}$ and the set of all positive integers that are divisible by 5 but not divisible by 3 consists of keeping 2 elements out of 3 of the former, we deduce that the density of the latter is $\frac{2}{3} \cdot \frac{1}{5} = \frac{2}{15}$.

(iv) Let us define $A_{\alpha,\beta} := \{\lfloor n\alpha + \beta \rfloor : n \in \mathbb{N}\}$ for $\alpha, \beta \geq 0$.

If $\alpha = 0$, then $A_{\alpha,\beta}$ consists of only one element, so $d(A_{\alpha,\beta}) = 0$.

If $0 < \alpha \leq 1$, one can easily show that $A_{\alpha,\beta}$ contains all natural numbers greater than β, so $d(A_{\alpha,\beta}) = 1$.

For $\alpha > 1$, notice that for any $n \in \mathbb{N}$ we have

$$\lfloor (n+1)\alpha + \beta \rfloor = \lfloor n\alpha + \beta + \alpha \rfloor \geq \lfloor n\alpha + \beta + 1 \rfloor = \lfloor n\alpha + \beta \rfloor + 1.$$

Hence, the sequence $(n\alpha + \beta)_{n\in\mathbb{N}}$ is strictly increasing. We can therefore use Proposition 54 to conclude that

$$d(A_{\alpha,\beta}) = \lim_{n\to\infty} \frac{n}{\lfloor n\alpha + \beta \rfloor} = \frac{1}{\alpha} \lim_{n\to\infty} \frac{n\alpha}{\lfloor n\alpha + \beta \rfloor} = \frac{1}{\alpha}.$$

Thus,

$$d(A_{\alpha,\beta}) = \begin{cases} 0, & \text{if } \alpha = 0; \\ 1, & \text{if } 0 < \alpha \leq 1; \\ \frac{1}{\alpha}, & \text{if } \alpha > 1. \end{cases}$$

E2. A homogeneous linear equation with integer coefficients $a_1, a_2, \ldots, a_r \in \mathbb{Z}\setminus\{0\}$ in the variables $x_1, \ldots, x_r$,

$$a_1x_1 + a_2x_2 + \cdots + a_rx_r = 0,$$

is said to *admit a distinct solution* if there exist distinct $x_1, \ldots, x_r \in \mathbb{N}$ satisfying the equation. The equation is called *density regular* if for every set $A \subset \mathbb{N}$ with positive upper density there exist distinct $x_1, \ldots, x_r \in A$ satisfying the equation.

Let $r \in \mathbb{N}$ and $a_1, a_2, \ldots, a_r \in \mathbb{Z} \setminus \{0\}$ and assume the equation $a_1x_1 + a_2x_2 + \cdots + a_rx_r = 0$ admits a distinct solution. Show that such an equation is density regular if and only if $a_1 + \cdots + a_r = 0$.

Hint: For the direction " $\Longrightarrow$ " use modular arithmetic and for the direction "$\Longleftarrow$" use Szemerédi's theorem.

Solution: ($\Longleftarrow$) Let $y_1, \ldots, y_r \in \mathbb{N}$ be a distinct solution of the equation and assume that

$$a_1 + \cdots + a_r = 0.$$

Define $k := \max_{1\leq i\leq r} y_i$ and let $A \subseteq \mathbb{N}$ be a subset with upper density. By Szemerédi's theorem, there exist $a, b \in \mathbb{N}$ such that

$$\{a + bj : 1 \leq j \leq k\} \subseteq A.$$

It follows that in particular

$$\{a + by_i : 1 \leq i \leq r\} \subseteq A.$$

This yields a solution in A for we have

$$\sum_{i=1}^{r} a_i(a + by_i) = a \underbrace{\sum_{i=1}^{r} a_i}_{=0} + b \underbrace{\sum_{i=1}^{r} a_iy_i}_{=0} = 0.$$

($\Longrightarrow$) Let $A := m\mathbb{N} + 1$, for some $m \in \mathbb{N}$. Clearly, A has positive upper density (it actually has density $\frac{1}{m}$). By density regularity there exist $y_1, \ldots, y_r \in \mathbb{N}$ such that

$$\sum_{i=1}^{r} a_i(my_i + 1) = 0.$$

Rearranging this equation yields

$$m \sum_{i=1}^{r} a_i y_i = -\sum_{i=1}^{r} a_i.$$

Therefore m divides $\sum_{i=1}^{r} a_i$. The only way this can hold for every $m \in \mathbb{N}$ is if $\sum_{i=1}^{r} a_i = 0$.

E3. Let $A \subseteq \mathbb{Z}_N$ with $|A| = \alpha N$. Prove the following are equivalent:

(i) $\sup_{\xi \neq 0} \left|\hat{A}(\xi)\right| = \alpha$;

(ii) there is an arithmetic progression $P = \left\{a, a+q, \ldots, a + \left(\frac{N}{q} - 1\right) q\right\}$ for some $a \in \mathbb{Z}_N$ and $q > 1$ with $q \mid N$ such that $A \subseteq P$.

Solution: We start with an observation. For $\xi \in \mathbb{Z}_N$, note that by the triangle inequality,

$$\left|\hat{A}(\xi)\right| = \left|\frac{1}{N} \sum_{n \in \mathbb{Z}_N} \mathbb{1}_A(n) e\left(-\frac{n\xi}{N}\right)\right| \leq \frac{1}{N} \sum_{n \in \mathbb{Z}_N} \left|\mathbb{1}_A(n) e\left(-\frac{n\xi}{N}\right)\right| = \alpha$$

with equality if and only if $e\left(-\frac{n\xi}{N}\right)$ is constant for $n \in A$.

(i) $\implies$ (ii). Let $\xi \in \mathbb{Z}_N$, $\xi \neq 0$, such that $\left|\hat{A}(\xi)\right| = \alpha$. By the above observation, there is a constant $c \in \mathbb{Z}_N$ such that $n\xi \equiv c \pmod{N}$ for every $n \in A$. The map $M_\xi : n \mapsto n\xi$ is a nonzero homomorphism from $\mathbb{Z}_N$ to $\mathbb{Z}_N$. Hence $\ker M_\xi$ is a proper subgroup of $\mathbb{Z}_N$, so it is of the form $\left\{0, q, 2q, \ldots, \left(\frac{N}{q} - 1\right) q\right\}$ for some $q > 1$ with $q \mid N$. Thus,

$$A \subseteq \{n \in \mathbb{Z}_N : M_\xi(n) = c\} = a + \ker M_\xi = \left\{a, a+q, \ldots, a + \left(\frac{N}{q} - 1\right) q\right\}$$

for any $a \in \mathbb{Z}_N$ with $M_\xi(a) = c$.

(ii) $\implies$ (i). Let $a \in \mathbb{Z}_N$ and $q > 1$ with $q \mid N$. Let $P = \left\{a, a+q, \ldots, a + \left(\frac{N}{q} - 1\right) q\right\}$, and suppose $A \subseteq P$. Then for $\xi = \frac{N}{q}$ and $n = a + dq \in A$, we have

$$e\left(-\frac{n\xi}{N}\right) = e\left(-\frac{a + dq}{q}\right) = e\left(-\frac{a}{q}\right).$$

Therefore,

$$\hat{A}(\xi) = \frac{1}{N} \sum_{n \in \mathbb{Z}_N} \mathbb{1}_A(n) e\left(-\frac{n\xi}{N}\right) = \frac{1}{N} \sum_{n \in A} e\left(-\frac{a}{q}\right) = e\left(-\frac{a}{q}\right) \alpha.$$

In particular, $\left|\hat{A}(\xi)\right| = \alpha$.