Exercise Set #9

Combinatorial Number Theory (2025)

E1. Compute the densities of the following sets:

(i) $\{17 + 8n : n \in \mathbb{N}\}$.

(ii) $\{2n + (-1)^n n : n \in \mathbb{N}\}$.

(iii) The set of all natural numbers that are divisible by 5 but not divisible by 3.

(iv) $\{\lfloor n\alpha + \beta \rfloor : n \in \mathbb{N}\}$ for $\alpha, \beta \geq 0$.

E2. A homogeneous linear equation with integer coefficients $a_1, a_2, \dots, a_r \in \mathbb{Z} \setminus \{0\}$ in the variables $x_1, \dots, x_r$,

$$a_1x_1 + a_2x_2 + \cdots + a_rx_r = 0,$$

is said to *admit a distinct solution* if there exist distinct $x_1, \dots, x_r \in \mathbb{N}$ satisfying the equation. The equation is called *density regular* if for every set $A \subset \mathbb{N}$ with positive upper density there exist distinct $x_1, \dots, x_r \in A$ satisfying the equation.

Let $r \in \mathbb{N}$ and $a_1, a_2, \dots, a_r \in \mathbb{Z} \setminus \{0\}$ and assume the equation $a_1x_1 + a_2x_2 + \cdots + a_rx_r = 0$ admits a distinct solution. Show that such an equation is density regular if and only if $a_1 + \cdots + a_r = 0$.

Hint: For the direction "$\implies$" use modular arithmetic and for the direction "$\impliedby$" use Szemerédi's theorem.

E3. Let $A \subseteq \mathbb{Z}_N$ with $|A| = \alpha N$. Prove the following are equivalent:

(i) $\sup_{\xi \neq 0} \left|\hat{A}(\xi)\right| = \alpha$;

(ii) there is an arithmetic progression $P = \left\{a, a+q, \dots, a + \left(\frac{N}{q} - 1\right) q\right\}$ for some $a \in \mathbb{Z}_N$ and $q > 1$ with $q \mid N$ such that $A \subseteq P$.