Exercise Set #8

Combinatorial Number Theory (2025)

E1. Show that Rado's theorem implies van der Waerden's theorem.

Solution: We have to show that for any $k \in \mathbb{N}$, k-term arithmetic progressions correspond to solutions to a system of equations satisfying the columns condition.

Let $k \in \mathbb{N}$ and suppose that $x_1, \ldots, x_k$ is an arithmetic progression. Equivalently, for any $i \in \{1, \ldots, k-2\}$, $x_i + x_{i+2} = 2x_{i+1}$. Hence the arithmetic progression corresponds to the equation $\mathbf{A}\mathbf{x} = \mathbf{0}$, where $x = (x_1\ x_2\ \ldots\ x_k)^{\mathrm{T}}$ and $A \in \mathbb{Z}^{(k-2)\times k}$ given by

$$A = \begin{pmatrix} 1 & -2 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \ldots & 0 & 0 & 0 & 0 \\ \\ 0 & 0 & 0 & 0 & \ldots & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -2 & 1 \end{pmatrix}.$$

Equivalently, $A = (a_{ij})$ with

$$a_{ij} = \begin{cases} 1, & \text{if } j = i \text{ or } j = i+2, \\ -2, & \text{if } j = i+1, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have that for all $i \in \{1, \ldots, k-2\}$,

$$\sum_{j=1}^{k} a_{ij} = a_{ii} + a_{i(i+1)} + a_{i(i+2)} = 0.$$

E2. Show that for any finite coloring of $\mathbb{N}$ there is one color class that contains solutions to all partition regular systems of linear equations.

Solution: **Hint:** Show first that if $\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$ and $\mathbf{B} \cdot \mathbf{y} = \mathbf{0}$ are partition regular, then

$$\begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \mathbf{0}$$

is partition regular.

It is straightforward that the matrix

$$\begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \end{pmatrix}$$

has the column property if A and B do, so by Rado's theorem the claim of the hint follows.

Now suppose that there is a finite coloring $\mathbb{N} = C_1 \cup \ldots \cup C_r$ such that for each i there is a partition regular system of equations $\mathbf{A}_i \cdot \mathbf{x} = \mathbf{0}$ which cannot be solved in C_i. It follows that the system $\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$, where $\mathbf{A} = \mathrm{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_r)$, is not partition regular. Applying the statement of the hint repeatedly it follows that there should be some $\mathbf{A}_i$ such that the system $\mathbf{A}_i \cdot \mathbf{x} = \mathbf{0}$ is not partition regular, but this is a contradiction.

E3. Given a set $A \subset \mathbb{N}$, we let $\mathrm{FS}(A)$ denote the set of all finite sums of A, that is,

$$\mathrm{FS}(A) = \left\{ \sum_{x \in F} x : \emptyset \neq F \subset A \text{ finite} \right\}.$$

Folkman's theorem: For any coloring of $\mathbb{N}$ and any $m \in \mathbb{N}$, there is a set $A \subset \mathbb{N}$ with $|A| = m$, such that $\mathrm{FS}(A)$ is monochromatic.

Show that Rado's theorem implies Folkman's theorem.

Solution: See https://joelmoreira.wordpress.com/2014/09/23/rados-theorem-and-deubers-theorem/.