Exercise Set #7
Combinatorial Number Theory (2025)

A set $S \subset \mathbb{N}$ is called:

i) *multiplicatively syndetic* if there are $n_1, \ldots, n_k \in \mathbb{N}$ such that $\mathbb{N} = S/n_1 \cup \ldots \cup S/n_k$,

ii) *multiplicatively thick* if for any finite set $F \subset \mathbb{N}$ there is $k \in \mathbb{N}$ such that $kF \subset S$,

iii) *multiplicatively piecewise syndetic* if there are $n_1, \ldots, n_k \in \mathbb{N}$ such that $S/n_1 \cup \ldots \cup S/n_k$ is multiplicatively thick (or equivalently, is the intersection of a multiplicatively syndetic and a multiplicatively thick set).

E1. i) Give an example of an additively syndetic set that is not multiplicatively syndetic.
ii) Give an example of a multiplicatively syndetic set that is not additively syndetic.

> ***Solution:*** i) $2\mathbb{N} + 1$ is additively syndetic, but clearly not multiplicatively syndetic.
> ii) Let $A = \bigcup_{k=1}^{\infty} [2^{2k}, 2^{2k+1})$. Then since for any $n \in \mathbb{N}$, either n or $2n$ in in A, it follows that A is multiplicatively syndetic. On the other hand it is easy to check that it is not additively syndetic.

E2. Show that if $S \subset \mathbb{N}$ is multiplicatively piecewise syndetic, then for each $k \in \mathbb{N}$ there are $a, d \in \mathbb{N}$ such that $\{d, a, a + d, a + 2d, \ldots, a + kd\} \subseteq S$.

> ***Solution:*** Let $n_1, \ldots, n_k \in \mathbb{N}$ such that $S/n_1 \cup \ldots \cup S/n_k$ is multiplicatively thick. Let $\ell \in \mathbb{N}$. By Brauers's theorem, we can pick $n \in \mathbb{N}$ such that whenever $\{1, \ldots, n\}$ is k-colored, there is monochromatic ℓ-AP which difference is of the same color. Since $S/n_1 \cup \ldots \cup S/n_k$ is multiplicatively thick, we can pick $m \in \mathbb{N}$ such that $m\{1, \ldots, n\} \subset S/n_1 \cup \ldots \cup S/n_k$. Then $\{1, \ldots, n\} \subset S/(n_1 m) \cup \ldots \cup S/(n_k m)$. Then there is $i \in \{1, \ldots, k\}$ and $a, d \in \mathbb{N}$ such that $\{d, a, a+d, \ldots, a+(\ell-1)d\} \in S/(n_i m)$, which implies that $\{dn_i m, an_i m, an_i m + dn_i m, \ldots, an_i m + (\ell - 1)dn_i m\} \in S$.

E3. Show that any multiplicatively syndetic set contains arbitrarily long geometric progressions.

> ***Solution:*** Let $k \in \mathbb{N}$. If S is mult. syn., then $S/1 \cup S/2 \cup \ldots \cup S/h$ covers the set $\{2^n : n \in \mathbb{N}\}$. We apply van der Waerden's theorem to $2^n : n \in \mathbb{N}$ to find $a, d \in \mathbb{N}$ such that $\{a, a + d, a + 2d, \ldots, a + kd\} \subseteq \{n \in \mathbb{N} : 2^n \in S/i\}$ for some i. In particular, we conclude that
> $$\{u, uv, uv^2, \ldots, uv^k\} \subseteq S$$
> where $u = i2^a, v = 2^d$.