Exercise Set #7
Combinatorial Number Theory (2025)

A set $S \subset \mathbb{N}$ is called:

i) *multiplicatively syndetic* if there are $n_1, \ldots, n_k \in \mathbb{N}$ such that $\mathbb{N} = S/n_1 \cup \ldots \cup S/n_k$,

ii) *multiplicatively thick* if for any finite set $F \subset \mathbb{N}$ there is $k \in \mathbb{N}$ such that $kF \subset S$,

iii) *multiplicatively piecewise syndetic* if there are $n_1, \ldots, n_k \in \mathbb{N}$ such that $S/n_1 \cup \ldots \cup S/n_k$ is multiplicatively thick (or equivalently, is the intersection of a multiplicatively syndetic and a multiplicatively thick set).

E1. i) Give an example of an additively syndetic set that is not multiplicatively syndetic.

ii) Give an example of a multiplicatively syndetic set that is not additively syndetic.

E2. Show that if $S \subset \mathbb{N}$ is multiplicatively piecewise syndetic, then for each $k \in \mathbb{N}$ there are $a, d \in \mathbb{N}$ such that $\{d, a, a+d, a+2d, \ldots, a+kd\} \subseteq S$.

E3. Show that any multiplicatively syndetic set contains arbitrarily long geometric progressions.