Exercise Set #6

Combinatorial Number Theory (2025)

E1. Provide a proof for the following properties that are mentioned in the lecture notes without a proof.

(i) $(A \cap B) - q = (A - q) \cap (B - q)$;

(ii) $(A \cup B) - q = (A - q) \cup (B - q)$;

(iii) $A^c - q = (A - q)^c$;

(iv) $A \subset B \implies A - q \subseteq B - q$.

> ***Solution:*** (i) Observe:
>
> $$\begin{aligned}(A \cap B) - n &= \{m \in \mathbb{N} : n + m \in A \cap B\} \\ &= \{m \in \mathbb{N} : n + m \in A \text{ and } n + m \in B\} \\ &= (A - n) \cap (B - n).\end{aligned} \tag{1}$$
>
> By definition we have that $(A \cap B) - q = \{n \in \mathbb{N} : (A \cap B) - n \in q\}$, so by (1),
>
> $$(A \cap B) - q = \{n \in \mathbb{N} : (A - n) \cap (B - n) \in q\}.$$
>
> On the other hand,
>
> $$\begin{aligned}(A - q) \cap (B - q) &= \{n \in \mathbb{N} : A - n \in q\} \cap \{n \in \mathbb{N} : B - n \in q\} \\ &= \{n \in \mathbb{N} : (A - n) \in q \text{ and } (B - n) \in q\}.\end{aligned}$$
>
> We therefore have to show:
>
> $$(A - n) \cap (B - n) \in q \iff (A - n) \in q \text{ and } (B - n) \in q.$$
>
> The direction $\implies$ holds because q is upward closed, and the direction $\impliedby$ holds because q is closed under finite intersections.
>
> (ii) Similarly to above, we have $(A \cup B) - n = (A - n) \cup (B - n)$, so
>
> $$(A \cup B) - q = \{n \in \mathbb{N} : (A - n) \cup (B - n) \in q\},$$
>
> while
>
> $$(A - q) \cup (B - q) = \{n \in \mathbb{N} : (A - n) \in q \text{ or } (B - n) \in q\}.$$
>
> If $(A - n) \cup (B - n) \in q$, then $(A - n) \in q$ or $(B - n) \in q$, since q is partition regular. On the other hand, if $(A - n) \in q$ or $(B - n) \in q$, then $(A - n) \cup (B - n) \in q$, since q is upward closed. Thus, $(A \cup B) - q = (A - q) \cup (B - q)$.
>
> (iii) We claim $A^c - n = (A - n)^c$ for $n \in \mathbb{N}$. This once again holds by a similar argument to part (i). Therefore,
>
> $$A^c - q = \{n \in \mathbb{N} : A^c - n \in q\} = \{n \in \mathbb{N} : (A - n)^c \in q\}.$$

By definition,

$$(A-q)^c = \{n \in \mathbb{N} : A-n \in q\}^c = \{n \in \mathbb{N} : A-n \notin q\}.$$

But for $n \in \mathbb{N}$,

$$(A-n)^c \in q \iff A-n \notin q$$

by Corollary 36 in the lecture notes, so $A^c - q = (A-q)^c$.

(iv) First, $A \subseteq B \implies A-n \subseteq B-n$ for all $n \in \mathbb{N}$. Therefore, if $A-n \in q$, then $B-n \in q$, since q is upward closed. It follows that

$$A-q = \{n \in \mathbb{N} : A-n \in q\} \subseteq \{n \in \mathbb{N} : B-n \in q\} = B-q.$$

E2. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a compact Hausdorff space X. Recall that $p\text{-}\lim_{n\in\mathbb{N}} x_n$ is defined by

$$x = p\text{-}\lim_{n\in\mathbb{N}} x_n \iff \forall\, U \subseteq X \text{ open neighborhood of } x, \{n \in \mathbb{N} : x_n \in U\} \in p.$$

Show that for any $p, q \in \beta\mathbb{N}$,

$$(p+q)\text{-}\lim_{n\in\mathbb{N}} x_n = p\text{-}\lim_{n\in\mathbb{N}} \big(q\text{-}\lim_{m\in\mathbb{N}} x_{n+m}\big).$$

In particular, if p is idempotent, then

$$p\text{-}\lim_{n\in\mathbb{N}} x_n = p\text{-}\lim_{n\in\mathbb{N}} \big(p\text{-}\lim_{m\in\mathbb{N}} x_{n+m}\big).$$

Solution: For $n \in \mathbb{N}$, let $y_n = q\text{-}\lim_{m\in\mathbb{N}} x_{n+m}$, and let $y = p\text{-}\lim_{n\in\mathbb{N}} y_n$. We want to show $(p+q)\text{-}\lim_{n\in\mathbb{N}} x_n = y$.

Let $U \subseteq X$ be an open neighborhood of y, and let $A = \{n \in \mathbb{N} : x_n \in U\}$. Note that

$$\begin{aligned} A \in p+q &\iff A-q \in p \\ &\iff \{n \in \mathbb{N} : A-n \in q\} \in p \\ &\iff \{n \in \mathbb{N} : \{m \in \mathbb{N} : m+n \in A\} \in q\} \in p \\ &\iff \{n \in \mathbb{N} : \{m \in \mathbb{N} : x_{n+m} \in U\} \in q\} \in p. \end{aligned} \tag{2}$$

By the definition of the limit along p, we have

$$\{n \in \mathbb{N} : y_n \in U\} \in p.$$

Let $n \in \mathbb{N}$ such that $y_n \in U$. Then by the definition of the limit along q, we have

$$\{m \in \mathbb{N} : x_{n+m} \in U\} \in q.$$

Therefore,

$$\{n \in \mathbb{N} : \{m \in \mathbb{N} : x_{n+m} \in U\} \in q\} \supseteq \{n \in \mathbb{N} : y_n \in U\}.$$

Since p is upward closed, we deduce

$$\{n \in \mathbb{N} : \{m \in \mathbb{N} : x_{n+m} \in U\} \in q\} \in p.$$

Thus, by (2), $A \in p+q$.

The set U was an arbitrary open neighborhood of y, so $(p+q)\text{-}\lim_{n\in\mathbb{N}} x_n = y$ by the definition of the limit along $p+q$.

E3. If A is an IP-set then for every $m \in \mathbb{N}$ the set $\{n \in A : m \mid n\}$ is an IP-set.

Solution: Let $A \subseteq \mathbb{N}$ be an IP-set and $x_1 < x_2 < ... \in \mathbb{N}$ such that $\mathrm{FS}(x_i)_{i=1}^{\infty} \subseteq A$.

For $j = 0, 1, ..., m-1$, let $C_j := \{n \in \mathbb{N} : n \equiv j \mod m\}$. Then

$$\mathbb{N} = C_0 \cup \cdots \cup C_{m-1}.$$

By the pigeonhole principle one of the cells, say C_k, contains infinitely many terms of the sequence. This yields

$$y_1 < y_2 < ... \in C_k,$$

such that $(y_i)_{i=1}^{\infty} \subseteq (x_i)_{i=1}^{\infty}$. Define $(z_i)_{i=1}^{\infty}$ by

$$\begin{aligned} z_1 &= y_1 + \ldots + y_m \\ z_2 &= y_{m+1} + \ldots + y_{2m} \\ &\vdots \\ z_n &= y_{(n-1)m+1} + \ldots + y_{nm} \\ &\vdots \end{aligned}$$

Then by construction, we have

- $z_1 < z_2 < ... \in \mathbb{N}$
- For any $n \in \mathbb{N}$, $z_n \equiv \underbrace{k + \cdots + k}_{m \text{ times}} \equiv km \equiv 0 \mod m$, i.e. $(z_n)_{n=1}^{\infty} \subseteq m\mathbb{N}$ and thus $\mathrm{FS}(z_i)_{i=1}^{\infty} \subset m\mathbb{N}$.
- $\mathrm{FS}(z_i)_{i=1}^{\infty} \subseteq \mathrm{FS}(y_i)_{i=1}^{\infty} \subseteq \mathrm{FS}(x_i)_{i=1}^{\infty} \subseteq A$ Indeed, a finite sum of z_i's yields a finite sum of y_i's.

Combining these results we get $\mathrm{FS}(z_i)_{i=1}^{\infty} \subseteq m\mathbb{N} \cap A = \{n \in A : m|n\}$, which shows that the latter is an IP-set.