Exercise Set Solutions #5

Combinatorial Number Theory (2025)

E1. Let p and q be ultrafilters on a set X. Show that if $A \cap B \neq \emptyset$ for all $A \in p$ and $B \in q$ then $p = q$.

> ***Solution:*** Assume $A \cap B \neq \emptyset$ for all $A \in p$ and $B \in q$. Define $r = \{A \cap B : A \in p,\ B \in q\}$. Note that r is non-empty (because it contains the set $\mathbb{N}$), does not contain the empty set (by assumption that $A \cap B \neq \emptyset$ for all $A \in p$ and $B \in q$), is upward closed (because p and q are upward closed), and closed under finite intersections (because p and q are closed under finite intersections). In conclusion, r satisfies the filter axioms and is therefore a filter. But $r \supset p$ and $r \supset q$ by definition of r. Since p and q are ultrafilters, they are maximal filters and so $p = r = q$, finishing the proof.

E2. Let p be an ultrafilter on $\mathbb{N}$. Show that p is non-principal if and only if it contains only infinite sets.

> ***Solution:*** Suppose that p contains only infinite sets. Then for every $n \in \mathbb{N}$, p does not contain the singleton $\{n\}$. Therefore $p \neq \delta_n$ (see Definition 37) and so p is non-principal. Conversely, if p is non-principal then for every $n \in \mathbb{N}$ we have $\{n\} \notin p$, otherwise by upward closeness, we would deduce that p is equal to δ_n. Note that any finite set can be partitioned into a finite union of singletons. By Proposition 35, we know that ultrafilters are partition regular. So if p were to contain a finite set, then it would contain a singleton, which is impossible as we have just seen. Therefore, p contains only infinite sets.

E3. Prove that there exists an ultrafilter on $\mathbb{N}$ every member of which is piecewise syndetic.

> ***Solution:*** The family $\mathcal{P}_{ps}^*$ is a filter:
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> - Clearly, $\emptyset \notin \mathcal{P}_{ps}^*$ and $\mathbb{N} \in \mathcal{P}_{ps}^*$.
> - Dual families are always upward closed.
> - By Corollary 26, $\mathcal{P}_{ps}$ is partition regular, so $\mathcal{P}_{ps}^*$ is closed under finite intersections by Proposition 20.
>
> Let $p \in \beta\mathbb{N}$ be an ultrafilter with $\mathcal{P}_{ps}^* \subseteq p$. Since p is an ultrafilter, we have $p = p^*$, so $p = p^* \subseteq \left(\mathcal{P}_{ps}^*\right)^* = \mathcal{P}_{ps}$.

E4. For a sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space X and a filter $\mathcal{F}$ on $\mathbb{N}$, we say $\mathcal{F}\text{-}\lim_{n\in\mathbb{N}} x_n = x$ if for every open neighborhood U of x in X, one has $\{n \in \mathbb{N} : x_n \in U\} \in \mathcal{F}$.

Prove: for any ultrafilter $p \in \beta\mathbb{N}$ and any sequence $(x_n)_{n\in\mathbb{N}}$ in a compact Hausdorff space X, $p\text{-}\lim_{n\in\mathbb{N}} x_n$ exists and is unique.
Hint: Do a proof by contradiction.

Solution: We will see that compactness guarantees existence, while Hausdorff guarantees uniqueness.

First we show existence. Suppose for contradiction that we have a sequence $(x_n)_{n\in\mathbb{N}}$ for which each $x \in X$ is not a limit along p. In particular this means that for every x there is a neighborhood U_x such that $\{n \in \mathbb{N} : x_n \in U_x\} \notin p$. Note that $\{U_x\}_{x\in X}$ is an open cover of X, and by compactness we have a finite subcover, say $U_{x^{(1)}}, \ldots, U_{x^{(k)}}$. For each $1 \leq j \leq k$ we have that $\{n \in \mathbb{N} : x_n \in U_{x^{(j)}}\} \notin p$. Since p is an ultrafilter this is equivalent to $\{n \in \mathbb{N} : x_n \in U_{x^{(j)}}\}^c = \{n \in \mathbb{N} : x_n \in U^c_{x^{(j)}}\} \in p$. Since $\bigcup_{j=1}^k U_{x^{(j)}} = X$ we deduce that $\bigcap_{j=1}^k U^c_{x^{(j)}} = \emptyset$. In particular this implies that

$$\bigcap_{j=1}^{k} \left\{n \in \mathbb{N} : x_n \in U^c_{x^{(j)}}\right\} = \left\{n \in \mathbb{N} : x_n \in \bigcap_{j=1}^{k} U^c_{x^{(j)}}\right\} = \emptyset.$$

But p is an ultrafilter, so it is closed under finite intersections. Hence, $\emptyset \in p$, a contradiction.

Now we show uniqueness. Suppose for contradiction that $p\text{-}\lim_{n\in\mathbb{N}} x_n = x$ and $p\text{-}\lim_{n\in\mathbb{N}} x_n = y$ with $x \neq y$. Since X is Hausdorff, there exist open neighborhoods U_x of x and U_y of y such that $U_x \cap U_y = \emptyset$. Since $p\text{-}\lim_{n\in\mathbb{N}} x_n = x$, we have $\{n \in \mathbb{N} : x_n \in U_x\} \in p$, and similarly, $\{n \in \mathbb{N} : x_n \in U_y\} \in p$. Since p is closed under finite intersection we have that

$$\{n \in \mathbb{N} : x_n \in U_x\} \cap \{n \in \mathbb{N} : x_n \in U_y\} \in p.$$

On the other hand,

$$\{n \in \mathbb{N} : x_n \in U_x\} \cap \{n \in \mathbb{N} : x_n \in U_y\} = \{n \in \mathbb{N} : x_n \in U_x \cap U_y\} = \emptyset,$$

and similarly as before we have therefore a contradiction.