Exercise Set #5

Combinatorial Number Theory (2025)

E1. Let p and q be ultrafilters on a set X. Show that if $A \cap B \neq \emptyset$ for all $A \in p$ and $B \in q$ then $p = q$.

E2. Let p be an ultrafilter on $\mathbb{N}$. Show that p is non-principal if and only if it contains only infinite sets.

E3. Prove that there exists an ultrafilter on $\mathbb{N}$ every member of which is piecewise syndetic.

E4. For a sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space X and a filter $\mathcal{F}$ on $\mathbb{N}$, we say $\mathcal{F}\text{-}\lim_{n\in\mathbb{N}} x_n = x$ if for every open neighborhood U of x in X, one has $\{n \in \mathbb{N} : x_n \in U\} \in \mathcal{F}$.

Prove: for any ultrafilter $p \in \beta\mathbb{N}$ and any sequence $(x_n)_{n\in\mathbb{N}}$ in a compact Hausdorff space X, $p\text{-}\lim_{n\in\mathbb{N}} x_n$ exists and is unique.
Hint: Do a proof by contradiction.