

Exercise Set Solutions #4

Combinatorial Number Theory (2025)

- E1.** For any finite coloring of \mathbb{N} and any $k \in \mathbb{N}$, show that there is a monochromatic k -term geometric progression (i.e., a monochromatic configuration of the form $\{a, ar, \dots, ar^{k-1}\}$ for some $a, r \in \mathbb{N}$, $r \geq 2$).

Solution: Let $\chi : \mathbb{N} \rightarrow \{1, \dots, r\}$ be a finite coloring and $k \in \mathbb{N}$. We define a new coloring $\tilde{\chi} : \mathbb{N} \rightarrow \{1, \dots, r\}$ by

$$\tilde{\chi}(n) = \chi(2^n).$$

By Van der Waerden's theorem, there exist $a, d \in \mathbb{N}$ such that the arithmetic progression $\{a, a + d, \dots, a + (k - 1)d\}$ is monochromatic with respect to $\tilde{\chi}$. Define $b = 2^a$ and $r = 2^d$. It follows then that $\{b, br, \dots, br^{k-1}\}$ is a geometric progression of length k which is monochromatic with respect to χ .

- E2.** Give an example of a finite coloring of \mathbb{N} without any monochromatic infinite arithmetic progression.

Solution: We define a red-blue coloring $\chi : \mathbb{N} \rightarrow \{\mathbf{r}, \mathbf{b}\}$ as shown in this table:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	...
$\chi(n)$	\mathbf{r}	\mathbf{b}	\mathbf{r}	\mathbf{r}	\mathbf{b}	\mathbf{b}	\mathbf{r}	\mathbf{r}	\mathbf{r}	\mathbf{b}	\mathbf{b}	\mathbf{b}	\mathbf{r}	\mathbf{r}	\mathbf{r}	\mathbf{r}	\mathbf{b}	\mathbf{b}	\mathbf{b}	\mathbf{b}	...

Let $(x_n)_{n \in \mathbb{N}}$ be an increasing monochromatic sequence. Then $\sup_{n \in \mathbb{N}}(x_{n+1} - x_n) = +\infty$ because the sequence must jump over the intervals of the opposite color and those have increasing length. It follows that there is no infinite monochromatic arithmetic progression for this coloring since such a sequence must have a constant difference between two consecutive terms.

- E3.** Show that in any finite coloring of \mathbb{N}^d , there are arbitrarily large monochromatic d -dimensional rectangular grids. That is, for any $k \in \mathbb{N}$, there is a monochromatic configuration of the form

$$\{(a_1 + i_1 n_1, \dots, a_d + i_d n_d) : 0 \leq i_1, \dots, i_d \leq k\}$$

for some $(a_1, \dots, a_d), (n_1, \dots, n_d) \in \mathbb{N}^d$.

Solution: Notice that any r -coloring $\xi : \mathbb{N}^d \rightarrow [r]$ of \mathbb{N}^d can be extended to a $r \cdot 2^d + 1$ coloring $\chi : \mathbb{Z}^d \rightarrow ([r] \times \{-1, 1\}^d) \cup \{0\}$ of \mathbb{Z}^d , where

$$\chi(x) = \begin{cases} 0 & \text{if } x_i = 0 \text{ for some } i \\ (j, i_1, \dots, i_d) & \text{if } \xi(|x_1|, \dots, |x_d|) = j \text{ and } i_j = x_j/|x_j|, \forall l \in [d]. \end{cases}$$

Using Gallai's theorem with the finite set $F = \{(i_1, \dots, i_d) \in \mathbb{N}^d : 0 \leq i_1, \dots, i_d \leq k\}$ we obtain $u \in \mathbb{Z}^d$ and $\lambda \in \mathbb{Z} \setminus \{0\}$ such that $u + \lambda F$ is monochromatic. Notice that $u + \lambda F$ must

lie in a quadrant of \mathbb{Z}^d . Moreover, by definition of χ , we notice that $(|u_1|, \dots, |u_d|) + |\lambda|F$ must be ξ -monochromatic, finishing the proof.

E4. A set is *AP-rich* if it contains a k -term arithmetic progression for every $k \in \mathbb{N}$. Show that the family \mathcal{P}_{AP} of AP-rich subsets of \mathbb{N} is partition regular.

Let $A \in \mathcal{P}_{AP}$, and suppose $A = \bigcup_{i=1}^r C_i$. By Proposition 31(ii), let $N = N(k, r) \in \mathbb{N}$ so that every r -coloring of $\{1, \dots, N\}$ has a monochromatic k -term arithmetic progression. Since $A \in \mathcal{P}_{AP}$, there is an N -term arithmetic progression

$$\{a, a + d, \dots, a + (N - 1)d\} \subseteq A.$$

Define an r -coloring $c : \{1, \dots, N\} \rightarrow \{1, \dots, r\}$ by $c(j) = i$ if $a + (j - 1)d \in C_i$. By the definition of N , there is a monochromatic k -term arithmetic progression

$$\{x, x + y, \dots, x + (k - 1)y\}$$

with respect to the coloring c . But then from the definition of c ,

$$\{a + (x - 1)d, a + (x - 1 + y)d, \dots, a + (x - 1 + (k - 1)y)d\} \subseteq C_i,$$

and this is a k -term arithmetic progression beginning at $a' = a + (x - 1)d$ with common difference $d' = yd$.

Thus, for any $k \in \mathbb{N}$, we can find a k -term arithmetic progression in one of the sets C_i . By the pigeonhole principle, it follows that one of the sets C_i is AP-rich, so \mathcal{P}_{AP} is partition regular.