Exercise Set Solutions #3

Combinatorial Number Theory (2025)

E1. Let $\mathcal{P}$ be a family of non-empty subsets of $\mathbb{N}$. Prove that if $\mathcal{P}$ is closed under finite intersections, then $\mathcal{P} \subset \mathcal{P}^*$.

> ***Solution:*** Suppose $A \subset \mathcal{P}$. Since $\mathcal{P}$ has the finite intersection property, it follows that for any $B \in \mathcal{P}$ we have $A \cap B \in \mathcal{P}$. Since elements in $\mathcal{P}$ are non-empty by assumption, we have $A \cap B \neq \emptyset$ and hence $A \in \mathcal{P}^*$. Since $A \subset \mathcal{P}$ was arbitrary, we have proven $\mathcal{P} \subset \mathcal{P}^*$.

E2. The *lower density* and *upper density* of a set of natural numbers A are defined respectively as

$$\underline{d}(A) = \liminf_{N\to\infty} \frac{|A \cap \{1,\dots,N\}|}{N} \qquad \text{and} \qquad \overline{d}(A) = \limsup_{N\to\infty} \frac{|A \cap \{1,\dots,N\}|}{N}.$$

Let

$$\mathcal{D} = \{A \subset \mathbb{N} : \overline{d}(A) > 0\}.$$

Show that if $A \in \mathcal{D}^*$, then A is thick.
Hint: It may be useful to find a description of the dual family $\mathcal{D}^*$.

> ***Solution:*** We claim that $\mathcal{D}$ is upward closed. Indeed, if $A \subseteq B \subseteq \mathbb{N}$ and $A \in \mathcal{D}$ then $B \in \mathcal{D}$ because
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> $$\overline{d}(B) = \limsup_{N\to\infty} \frac{|B \cap \{1,\dots,N\}|}{N} \geq \limsup_{N\to\infty} \frac{|A \cap \{1,\dots,N\}|}{N} = \overline{d}(A) > 0.$$
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> Hence we can use the fact that $A \in \mathcal{D}^*$ if and only if $A^c \notin \mathcal{D}$ and we deduce that
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> $$\mathcal{D}^* = \{A \subseteq \mathbb{N} : \overline{d}(A^c) = 0\}.$$
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> We note also that $\overline{d}(A^c) = 0 \iff \underline{d}(A) = 1$, indeed if $\underline{d}(A) = 1$ then $\overline{d}(A) = 1$, and we have that
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> $$0 \leq \overline{d}(A^c) = \limsup_{N\to\infty} \frac{|A^c \cap \{1,\dots,N\}|}{N} \leq 1 - \limsup_{N\to\infty} \frac{|A \cap \{1,\dots,N\}|}{N} = 0$$
>
> on the other hand if $\overline{d}(A^c) = 0$ in particular also $\underline{d}(A^c) = 0$ and
>
> $$1 \geq \underline{d}(A) = \liminf_{N\to\infty} \frac{|A \cap \{1,\dots,N\}|}{N} \geq 1 - \limsup_{N\to\infty} \frac{|A^c \cap \{1,\dots,N\}|}{N} = 1$$
>
> so
>
> $$\mathcal{D}^* = \{A \subseteq \mathbb{N} : \underline{d}(A) = 1\}.$$
>
> Now we show that $A \in \mathcal{D}^*$ implies A is thick. Since $\underline{d}(A) = 1$, we have $\underline{d}(A-1) = \underline{d}(A-2) = \ldots = \underline{d}(A-(h-1)) = 1$. Note that the intersection of two sets with lower density 1 is a set with lower density 1. Therefore, the intersection of finitely many sets with density 1 is a set with density 1. We conclude that
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> $$\underline{d}(A \cap (A-1) \cap \ldots \cap (A-h+1)) = 1.$$

In particular, the intersection $A \cap (A-1) \cap \ldots \cap (A-h+1)$ is non-empty, which means there exists some element $a \in A \cap (A-1) \cap \ldots \cap (A-h+1)$. The interval $\{a, a+1, a+2, \ldots, a+h-1\}$ is a subset of A, finishing the proof.

E3. **(a)** Show that the family of sets $\Sigma = \{A \subseteq \mathbb{N} : \exists B \subseteq \mathbb{N} \text{ infinite such that } B \oplus B \subseteq A\}$ is partition regular.

Solution: First of all, we show that Σ is partition regular. Clearly, we have that Σ is upward closed, hence it will suffice to show that if $A \in \Sigma$ and $A = A_1 \cup A_2$ with $A_1 \cap A_2 = \emptyset$ then either $A_1 \in \Sigma$ or $A_2 \in \Sigma$. Since $A \in \Sigma$ there exists $B \subseteq \mathbb{N}$ infinite such that $B \oplus B \subseteq A$. We define a coloring $\chi : B^{(2)} \to \{1, 2\}$, defined by $\chi(\{b_1, b_2\}) = j$ if and only if $b_1 + b_2 \in A_j$ for $j = 1, 2$. By Ramsey's Theorem, we have that there exists an infinite set $C \subseteq B$ such that $C^{(2)}$ is monochromatic, meaning that $C \oplus C \subseteq A_j$ for $j = 1$ or $j = 2$. We conclude that either $A_1 \in \Sigma$ or $A_2 \in \Sigma$.

(b) Show that Σ contains all thick sets.

Solution: Next, we claim that thick sets belong to Σ. Let $T \subseteq \mathbb{N}$ be thick. Pick $b_1 \in T$. Since T is thick, there is an interval $\{b_2, b_2 + 1, \ldots, b_2 + b_1\} \subseteq T$ for some $b_2 \in \mathbb{N}$. Suppose we have chosen $b_1 < \cdots < b_n \in T$ with $b_i + b_j \in T$ for $1 \leq i < j \leq n$. Then there exists $b_{n+1} \in \mathbb{N}$ such that $\{b_{n+1}, b_{n+1} + 1, \ldots, b_{n+1} + b_n\} \subseteq T$. Note that $b_{n+1} + 1 \leq b_i + b_{n+1} \leq b_{n+1} + b_n$ for $1 \leq i \leq n$, so $b_i + b_{n+1} \in T$. The set $B = \{b_1 < b_2 < \ldots\}$ then satisfies $B \oplus B \subseteq T$.

(c) Let $A \subseteq \mathbb{N}$ be a piecewise syndetic set. Show that there are infinite sets $B, C \subseteq \mathbb{N}$ such that

$$B + C = \{b + c : b \in B, c \in C\} \subseteq A.$$

Hint: Prove the stronger statement that a shift of A contains a set of the form $B \oplus B = \{b_1 + b_2 : b_1, b_2 \in B, b_1 \neq b_2\}$ for some infinite set $B \subseteq \mathbb{N}$. (Why is this stronger?)

Solution: Now if A is piecewise syndetic then we get that there exists a finite set F such that $\bigcup_{n \in F}(A - n)$ is thick. Hence, $\bigcup_{n \in F}(A - n) \in \Sigma$. Since Σ is partition regular, there exists $n \in F$ such that $A - n \in \Sigma$. Let $X \subseteq \mathbb{N}$ be an infinite set such that $X \oplus X \subseteq A - n$. Let $X_1, X_2 \subseteq X$ be disjoint infinite sets. Let $B = X$ and $C = X + n$. Then for any $b \in B$ and $c \in C$, we have $b + c \in X + X + n \subseteq (X \oplus X) + n \subseteq A$, so $B + C \subseteq A$ as desired.