# Exercise Set #3
## Combinatorial Number Theory (2025)

---

**E1.** Let $\mathcal{P}$ be a family of non-empty subsets of $\mathbb{N}$. Prove that if $\mathcal{P}$ is closed under finite intersections, then $\mathcal{P} \subset \mathcal{P}^*$.

**E2.** The *lower density* and *upper density* of a set of natural numbers $A$ are defined respectively as

$$\underline{d}(A) = \liminf_{N\to\infty} \frac{|A \cap \{1,\ldots,N\}|}{N} \qquad \text{and} \qquad \overline{d}(A) = \limsup_{N\to\infty} \frac{|A \cap \{1,\ldots,N\}|}{N}.$$

Let

$$\mathcal{D} = \{A \subset \mathbb{N} : \overline{d}(A) > 0\}.$$

Show that if $A \in \mathcal{D}^*$, then $A$ is thick.
**Hint:** It may be useful to find a description of the dual family $\mathcal{D}^*$.

**E3.** **(a)** Show that the family of sets $\Sigma = \{A \subseteq \mathbb{N} : \exists B \subseteq \mathbb{N} \text{ infinite such that } B \oplus B \subseteq A\}$ is partition regular.

**(b)** Show that $\Sigma$ contains all thick sets.

**(c)** Let $A \subseteq \mathbb{N}$ be a piecewise syndetic set. Show that there are infinite sets $B, C \subseteq \mathbb{N}$ such that

$$B + C = \{b + c : b \in B, c \in C\} \subseteq A.$$

**Hint:** Prove the stronger statement that a shift of $A$ contains a set of the form $B \oplus B = \{b_1 + b_2 : b_1, b_2 \in B, b_1 \neq b_2\}$ for some infinite set $B \subseteq \mathbb{N}$. (Why is this stronger?)