Exercise Set Solutions #2
Combinatorial Number Theory (2025)

E1. Let $K \subset \mathbb{R}^2$ be infinite. Show that either there exists an infinite subset of K so that no four points lie on the same circle, or there is an infinite subset of K whose points all lie on the same circle.

> ***Solution:*** We define a coloring $\chi : K^{(4)} \to \{C_1, C_2\}$ by
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> $$\chi(\{x,y,z,w\}) = \begin{cases} C_1 & \text{if } x,y,z,w \text{ do not lie on a circle} \\ C_2 & \text{if } x,y,z,w \text{ all lie on a circle.} \end{cases}$$
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> By Ramsey's theorem for 4-sets, there exist an infinite subset $M \subseteq K$ such that $M^{(4)}$ is monochromatic.
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> If $M^{(4)}$ is of color C_1, no quadruplet of M lie on the same circle, hence we are done.
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> If $M^{(4)}$ is of color C_2 all quadruplet of M lie on a circle, we still have to prove that they share the same circle. Consider an element $\{x,y,z,w\}$ of $M^{(4)}$, there exist a unique circle C such that x,y,z lie on this circle, this is the circumscribed circle of the triangle xyz, (actually, this circle would not exist if x,y and z were aligned, but then $\{x,y,z,w\}$ would not be colored in C_2). Now since C is unique, w must lie on C for the assumption $\{x,y,z,w\} \in M^{(4)}$ to hold. Since for fixed $x,y,z \in M$, we have : $\forall w \in M, \{x,y,z,w\} \in M^{(4)}$, this reasoning holds for any w, and each of them must lie on C, therefore all points of M lie on C. M is infinite, hence we are done.

E2. Show that for any function $f \colon \mathbb{N} \to \mathbb{N}$ there exists a 3-coloring of $\mathbb{N}$ such that for any $x \in \mathbb{N}$ we have

$$\{x, f(x)\} \text{ is monochromatic} \iff x = f(x).$$

> ***Solution:*** Define the coloring $c_1 : \mathbb{N} \to \{1,2,3,4\}$ (where 1 represents red, 2 blue, 3 green, and 4 white) such that 1 is red, and for each $n \in \mathbb{N}$
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> - If all the elements in $\{f^n(1)\}_{n\geq 0}$ are different, color $f^n(1)$ red if n is even, and blue if not, where $f^n = f \circ \cdots \circ f$ is the n-composition of f.
> - If there is $n \in \mathbb{N}$ such that $f^n(1) = f^j(1)$ for $j \leq n-1$, then the sequence $\{f^n(1)\}_{n\geq 0}$ gets eventually stuck in a loop. In this case, if $n \in \mathbb{N}$ is the minimum such that that happens, for $j \leq n-2$ we color $f^j(1)$ red if j is even and blue if it is odd. We color $f^{n-1}(1)$ green to avoid breaking the rule.
> - If there is $n \in \mathbb{N}$ such that $f^n(1)$ is the same as $f^{n-1}(n)$, we color them the same color. If $n \in \mathbb{N}$ is the minimum such that this happens, we color $j \leq n$ in the same manner as in the previous steps, i.e. red if j is even and blue otherwise.
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> Also, we color $\mathbb{N} \setminus \{f^n(1)\}_{n\geq 0}$ white. Thus, we obtain a coloring $c_1 \in [4]^{\mathbb{N}}$ such that the desired condition is fulfilled in $\{f^n(1)\}_{n\geq 0}$ and the rest is colored with white. Assume that

we have defined a 4-coloring c_k of $\mathbb{N}$ such that the the desire condition is fulfill in

$$S(k) := \bigcup_{i=1}^{k} \{f^n(i)\}_{n\geq 0}.$$

We define a coloring c_{k+1} as follows: for $n \in S(k)$, $c_{k+1}(n) = c_k(n)$. We divide in cases:

- If $k+1 \in S(k)$ then we are done, as $\{f^n(k+1)\}_{n\geq 0} \subseteq S(k)$.
- If not, further assume that there is $n \in \mathbb{N}$ such that $f^n(k+1) \in S(k)$. Without loss of generality, take $n \in \mathbb{N}$ the minimum satisfying such condition. Then, using the two colors in $\{1,2,3\} \setminus \{c_k(f^n(k+1))\}$ we color the sequence $(k+1), f(k+1), ..., f^{n-1}(k+1)$ interspersing those two colors.
- Finally, if there is no $n \geq 0$ such that $f^n(k+1) \in S(k)$, we repeat the coloring of the base case for $\{f^n(k+1)\}_{n\geq 0}$.

The rest of the numbers are colored white. Thus, we obtain a coloring c_{k+1} such that the condition is fulfilled in $S(k+1)$. Since the space $[4]^{\mathbb{N}}$ is compact, the sequence $(c_k)_{k\in\mathbb{N}}$ has an accumulation point (actually, it converges). Call c such an accumulation point. Clearly c fulfills the desired condition in

$$\bigcup_{i=1}^{\infty} \{f^n(i)\}_{n\geq 0} = \mathbb{N},$$

and it does not use the color white, so it is a coloring in 3 colors, finishing.

E3. We color $\mathbb{Z}^d$ using finitely many colors. Show that for any $2 \leq r \leq d$ there exists a monochromatic r-dimensional rectangle.

Solution: Suppose $\mathbb{Z}^d$ is colored using at most m colors. Let $R = R_d(2d, m)$ denote the d-uniform Ramsey number for $(2d, m)$. Let $K_R^{(d)}$ denote the complete d-uniform graph on the vertices $\{v_1, \ldots, v_R\}$. For any $i_1, \ldots, i_d \in \{1, \ldots, R\}$ with $i_1 < \cdots < i_d$ we can assign to the edge $\{v_{i_1}, \ldots, v_{i_d}\}$ of $K_R^{(d)}$ the same color as the color given to the point $(i_1, \ldots, i_d)$ in $\mathbb{Z}^d$. This induces a coloring of $K_R^{(d)}$ using at most m colors. By Ramsey's Theorem for hypergraphs, there exist $i_1 < i_2 < \cdots < i_{2d} \in \{1, \ldots, R\}$ such that the complete graph on the vertices $\{v_{i_1}, v_{i_2}, \ldots, v_{2d}\}$ is monochromatic. In particular, the 2^d edges

$$\{v_{i_{j_1}}, v_{j_2}, \ldots, v_{j_d} : j_1 < \cdots < j_d \leq 2d\}$$

have the same color. This implies that the four lattice points

$$(x_1, \ldots, x_d) \text{ such that } x_j \in \{i_j, i_{d+j}\} \text{ for each } j \in [d]$$

have the same color, yielding a monochromatic rectangle as desired.

E4. Given any collection of real numbers $x_1, \ldots, x_{mnk+1} \in \mathbb{R}$, prove that there is either a strictly increasing subsequence of length $m+1$, a strictly decreasing subsequence of length $n+1$, or a constant subsequence of length $k+1$.

Solution: Start by counting the number of appearances of each distinct value in the sequence. If some value occurs at least $k+1$ times, then we have found a constant subsequence of length $k+1$ and we are done. If no value appears more than k times, then there are at least $\lceil \frac{mnk+1}{k} \rceil = mn+1$ distinct values in our sequence, say $x_{i_1}, x_{i_2}, \ldots, x_{i_{mn+1}}$. Restricting to this subsequence and applying the Erdős–Szekeres Theorem on monotone paths, we are guaranteed the existence of either a strictly increasing sequence of length $m+1$ or a strictly decreasing sequence of length $n+1$, as desired.