Exercise Set #2

Combinatorial Number Theory (2025)

E1. Let $K \subset \mathbb{R}^2$ be infinite. Show that either there exists an infinite subset of K so that no four points lie on the same circle, or there is an infinite subset of K whose points all lie on the same circle.

E2. Show that for any function $f\colon \mathbb{N} \to \mathbb{N}$ there exists a 3-coloring of $\mathbb{N}$ such that for any $x \in \mathbb{N}$ we have

$$\{x, f(x)\} \text{ is monochromatic} \quad \Longleftrightarrow \quad x = f(x).$$

E3. We color $\mathbb{Z}^d$ using finitely many colors. Show that for any $2 \leq r \leq d$ there exists a monochromatic r-dimensional rectangle.

E4. Given any collection of real numbers $x_1, \ldots, x_{mnk+1} \in \mathbb{R}$, prove that there is either a strictly increasing subsequence of length $m+1$, a strictly decreasing subsequence of length $n+1$, or a constant subsequence of length $k+1$.