# Exercise Set Solutions #1

## Combinatorial Number Theory (2025)

---

**E1.** Show that for any $n \in \mathbb{N}$ there exists a number $C = C(n)$ such that any graph with at least $C$ vertices contains either a clique[1] of size $n$ or an independent set[2] of size $n$.

> ***Solution:*** Take $C(n) = R(n, 2)$, i.e. the ramsey number for 2 coloring and Clique of $n$ vertices. Consider a graph $(V, E(V))$ of $C$ vertices, and now consider the 2-coloring of $(K_C, E(K_C))$ given by
>
> $$e \in E(K_C) \text{ is red } \iff e \in E(V).$$
>
> By the Ramsey theorem, there is a monochromatic clique in $(K_C, E(K_C))$. If such clique is red, then it is a subgraph of $(V, E(V))$ and we found a clique in it. If such clique is not red (let's say blue), then no of the edges of such clique is in $E(V)$, which implies that such clique is an independent set of size $n$ in $(V, E(V))$, concluding that we have one of the two cases.

**E2.** Prove that every infinite sequence of real numbers has an infinite monotonic subsequence.
*Remark*: This can be used to give a short proof of the Bolzano–Weierstrass theorem: every bounded sequence of real numbers has a convergent subsequence.

> ***Solution:*** Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Define a 2-coloring of $\mathbb{N}^{(2)}$ by
>
> $$c(i,j) = \begin{cases} \texttt{increase}, & \text{if } x_{\min\{i,j\}} \leq x_{\max\{i,j\}} \\ \texttt{decrease}, & \text{if } x_{\min\{i,j\}} > x_{\max\{i,j\}}. \end{cases}$$
>
> By Ramsey's theorem for 2-sets, there is an infinite set $Y \subseteq \mathbb{N}$ such that $Y^{(2)}$ is monochromatic with respect to $c$. Enumerate $Y = \{n_1 < n_2 < \dots\}$. If $Y^{(2)}$ is monochromatic in the color `increase`, then $x_{n_i} \leq x_{n_j}$ for every $i < j$. Therefore, $(x_{n_k})_{k\in\mathbb{N}}$ is monotonically nondecreasing. If, on the other hand, $Y^{(2)}$ is monochromatic in the color `decrease`, then $x_{n_i} > x_{n_j}$ for every $i < j$, so $(x_{n_k})_{k\in\mathbb{N}}$ is a decreasing sequence.

**E3.** Prove that any finite coloring of $\mathbb{N}$ admits a monochromatic solution to the equation $xy = z$ with $x, y \geq 2$.

> ***Solution:*** Let $\chi : \mathbb{N} \to \{1, \dots, r\}$ be any given finite coloring of $\mathbb{N}$. We construct a new coloring $\tilde{\chi} : \mathbb{N} \to \{1, \dots, r\}$ defined by $\tilde{\chi}(n) = \chi(2^n)$. Applying Schur's Theorem we have that there exists $n, m, k \in \mathbb{N}$ which are $\tilde{\chi}$-monochromatic and such that $n + m = k$. By construction, we have that $x = 2^n$, $y = 2^m$, and $z = 2^k$ gives a $\chi$-monochromatic solution to $xy = z$ with $x, y \geq 2$.

**E4.** **(a)** Show that any edge-coloring of $K_6$ with 2 colors admits at least 2 monochromatic triangles.

---

[1] A subset of vertices of a graph $G = (V, E)$ is called a *clique* if every two distinct vertices in the clique are connected by an edge. In other words, a clique is a copy of a complete graph.

[2] A subset of vertices of $G$ is called an *independent set* if no two vertices are connected by an edge. Independent sets are sometimes also referred to as *anticliques*.

**(b)** Show that any edge-coloring of $K_7$ with 2 colors admits at least 4 monochromatic triangles.

***Solution:*** A preliminary remark: In what follows, we will denote a triangle by a 3-set of vertices. When talking about a coloring of a triangle, one should bear in mind that we are making reference to the coloring of the edges.

First we show that any edge-coloring of $K_6$ with 2 colors admits at least 2 monochromatic triangles ($\star$): Let us consider a blue/red edge-coloring of $K_6$. Since $R(3,2) = 6$, we know that there exists at least one monochromatic triangle $T_1$ that we may assume to be blue (otherwise flip the colors). Let us label the vertices of this triangle $v_1, v_2, v_3$.

Consider now the triangle induced by the three other vertices.

- If it is monochromatic blue, we are done.
- Otherwise one edge must be red. Denote the two vertices incident to this red edge by $v_4, v_5$ and consider the three edges between $v_4$ and vertices in $T_1$, namely $\{v_4, v_1\}$, $\{v_4, v_2\}$ and $\{v_4, v_3\}$.
  - If at least two of them are blue (for example $\{v_4, v_1\}$ and $\{v_4, v_2\}$), we have found another blue monochromatic triangle (in this example $\{v_1, v_2, v_4\}$) and we are done.
  - Else two of them must be red. To help us visualise the situation, we will assume without loss of generality that these edges are $\{v_4, v_1\}$ and $\{v_4, v_2\}$ because we could simply switch the labelling between $v_1, v_2$ and $v_3$. Now we use the same argument with $v_5$ instead of $v_4$. Hence we consider the three edges between $v_5$ and the vertices in $T_1$, i.e., the edges $\{v_5, v_1\}$, $\{v_5, v_2\}$ and $\{v_5, v_3\}$.
    - If at least two of them are blue, like before we have found another blue monochromatic triangle and we are done.
    - Else two of them must be red. Therefore either $\{v_5, v_1\}$ or $\{v_5, v_2\}$ is red, leading to the existence of a red monochromatic triangle. (For instance if $\{v_5, v_1\}$ is red, $\{v_1, v_4, v_5\}$ is monochromatic red). Therefore we have a second monochromatic triangle.

Let us now consider a blue/red edge-coloring of $K_7$. Removing one vertex (and all the edges containing it) leaves a subgraph which is a copy of $K_6$. By applying ($\star$) to this copy of $K_6$, we get two monochromatic triangles $T_1, T_2$. Two cases might occur:

- $T_1$ and $T_2$ share at least a vertex $v$. In this case, remove this vertex from $K_7$, which gives another copy of $K_6$ and apply the same argument again to get two other monochromatic triangles $T_3$ and $T_4$. Since they do not contain $v$ they are both distinct from $T_1$ and $T_2$, hence we have a total of four monochromatic triangles in $K_7$.
- $T_1$ and $T_2$ are disjoint. In this case, we label the vertices of $K_7$ such that $T_1 = \{v_1, v_2, v_3\}$ and $T_2 = \{v_4, v_5, v_6\}$. By applying ($\star$) to the copy of $K_6$ obtained by removing $v_1$ we get two monochromatic triangles that are not $T_1$, but since one of them could be $T_2$ we are only guaranteed to have a third triangle $T_3$ distinct from $T_1$ and $T_2$. By the pigeonhole principle, $T_3$ must intersect either $\{v_2, v_3\}$ or $\{v_4, v_5, v_6\}$.
  - If it intersects $\{v_2, v_3\}$, say $v_2$ for concreteness, then by applying ($\star$) to the copy of $K_6$ obtained by removing $v_2$ we get two monochromatic triangles that are neither

$T_1$ nor $T_3$. One of them might be $T_2$, but regardless we obtain a fourth distinct monochromatic triangle $T_4$.

- Similarly if it intersects $\{v_4, v_5, v_6\}$, say $v_4$ for concreteness, the by applying $(\star)$ to the copy of $K_6$ obtained by removing $v_4$ we get two monochromatic triangles that are neither $T_2$ nor $T_3$. One of them might be $T_1$ but we obtain a fourth distinct monochromatic triangle $T_4$.

In any case, we have at least four monochromatic triangles in $K_7$.