Exercise Set #1

Combinatorial Number Theory (2025)

E1. Show that for any $n \in \mathbb{N}$ there exists a number $C = C(n)$ such that any graph with at least C vertices contains either a clique[1] of size n or an independent set[2] of size n.

E2. Prove that every infinite sequence of real numbers has an infinite monotonic subsequence.
Remark: This can be used to give a short proof of the Bolzano–Weierstrass theorem: every bounded sequence of real numbers has a convergent subsequence.

E3. Prove that any finite coloring of $\mathbb{N}$ admits a monochromatic solution to the equation $xy = z$ with $x, y \geq 2$.

E4. **(a)** Show that any edge-coloring of K_6 with 2 colors admits at least 2 monochromatic triangles.

(b) Show that any edge-coloring of K_7 with 2 colors admits at least 4 monochromatic triangles.

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[1] A subset of vertices of a graph $G = (V, E)$ is called a *clique* if every two distinct vertices in the clique are connected by an edge. In other words, a clique is a copy of a complete graph.

[2] A subset of vertices of G is called an *independent set* if no two vertices are connected by an edge. Independent sets are sometimes also referred to as *anticliques*.