# Exercise Set Solutions #14

## Combinatorial Number Theory (2025)

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**E1.** Decide if the following statement is true: for all sets $A, B \subseteq \mathbb{N}$ with upper Banach densities equal to 1, we have that $A - B$ is syndetic.

> ***Solution:*** The statement is false. Set the sets
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> $$A = \bigcup_{n=1}^{\infty} \{(2n)! - n, ..., (2n)! + n], \quad B = \bigcup_{n=1}^{\infty} \{(2n+1)! - n, ..., (2n+1)! + n\}.$$
>
> Clearly $A$ and $B$ have upper Banach density 1 as they contain arbitrarily large intervals. We also observe that $A - B$ does not contain the intervals of the form $\{(2n-1)1 + n + 1, \ldots, (2n)! - n\}$ which have arbitrarily length. Thus, $A - B$ is not syndetic.

**E2.** Let $n, m \in \mathbb{N}$, $G \subseteq [m]$ and $H \subseteq [n]$. Show that there exists $z \in \mathbb{N}$ such that

$$\frac{|(G - z) \cap H|}{n} \geq \frac{|G|}{m} \cdot \frac{|H|}{n} - \frac{|H|}{m}$$

> ***Solution:*** We have that
>
> $$\begin{aligned} \sum_{x=1}^{m} \left( \sum_{h \in H} \mathbb{1}_G(x+h) \right) &= \sum_{h \in H} \left( \sum_{x=1}^{m} \mathbb{1}_G(x+h) \right) \\ &= \sum_{h \in H} |G \cap \{h+1, ..., m\}| \\ &= \sum_{h \in H} (|G| - h) \geq |H|(|G| - n). \end{aligned}$$
>
> By the pigeonhole principle, there must be at least one $z$ such that
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> $$\frac{|(G - z) \cap H|}{n} = \frac{1}{n} \sum_{h \in H} \mathbb{1}_C(z+h) \geq \frac{1}{n} \frac{|H| \cdot (|G| - n)}{m} = \frac{|G|}{m} \frac{|H|}{n} - \frac{|H|}{m}.$$

**E3.** For $n \in \mathbb{N}$, let $E_n \subseteq [n]$. Assume that $\lim_{n \to \infty} |E_n|/n = \gamma > 0$. We will show that there exists a finite set $F$ of cardinality $|F| \leq 1/\gamma$ such that for every $m$ :

$$[m] \subseteq (E_n - E_n) + F, \quad \text{for infinitely many } n. \tag{1}$$

- **(a)** Set $m_0 = 0$, $F = \{m_0\}$ $\Gamma_0 = \mathbb{N}$, $\Lambda_0 = \emptyset$. Assume that $(F_{i-1}, m_{i-1}, \Gamma_{i-1}, \Lambda_{i-1})$ were defined. Construct $(F_i, m_i, \Gamma_i, \Lambda_i)$ following the next process: If for every $m \in \mathbb{N}$ the set

  $$\{n \in \Gamma_i : [m] \subseteq (E_n - E_n) + F_i\}$$

  is infinite we stop. Otherwise, set $m_{i+1}$ as the minimal $m \in \mathbb{N}$ for which (1) does not hold and set $F_{i+1} = \{m_1, \ldots, m_{i+1}\}$,

  $$\Gamma_{i+1} = \{n \in \Gamma_i : [m_{i+1} - 1] \subseteq (E_n - E_n) + F_{i+1}\}$$

and

$$\Lambda_{i+1} = \{n \in \mathbb{N} : m_{i+1} \in (E_n - E_n) + F_{i+1}\}.$$

Show that for each $i \in \mathbb{N}$, for which $(F_i, m_i, \Gamma_i, \Lambda_i)$ were defined, if $N \in \Gamma_i \setminus (\Lambda_1 \cup \cdots \cup \Lambda_i)$, then

$$\frac{|E_N|}{N} \leq \frac{1}{i} + \frac{m_i}{iN}.$$

> ***Solution:*** We start observing that each $\Gamma_j$ is infinite, and each $\Lambda_j$ is finite. Thus, the set $X_i := \Gamma_i \setminus (\Lambda_1 \cup \cdots \cup \Lambda_i)$ is infinite. Observe that for $N \in X_i$ the sets in the family $\{E_N + m_j | j = 1, \ldots, j\}$ are pairwise disjoint. Now, every $E_N + m_j \subseteq [N + m_j]$, so we obtain
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> $$N + m_j \geq \left| \bigcup_{j=1}^{i} (E_N + m_j) \right| = \sum_{j=1}^{i} |(E_N + m_j)| = i|E_N|,$$
>
> which implies that
>
> $$\frac{|E_N|}{N} \leq \frac{1}{i} + \frac{m_i}{iN}.$$

**(b)** Show that the previous process must stop at step $i \leq 1/\gamma$. Conclude.

> ***Solution:*** We can take $N$ approaching to infinite in
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> $$\frac{|E_N|}{N} \leq \frac{1}{i} + \frac{m_i}{iN},$$
>
> getting that $\gamma \leq 1/i$. This implies that the previous process must stop at some step $i \leq 1/\gamma$. Finally, we observe by definition of $F_i$ and $m_1, \ldots, m_i$, for each $n \in \Gamma_i$ and each $m \in \mathbb{N}$ we have the inclusion $[m] \subseteq (E_n - E_n) + F_i$, finishing.