Exercise Set #14

Combinatorial Number Theory (2025)

E1. Decide if the following statement is true: for all sets $A, B \subseteq \mathbb{N}$ with upper Banach densities equal to 1, we have that $A - B$ is syndetic.

E2. Let $n, m \in \mathbb{N}$, $G \subseteq [m]$ and $H \subseteq [n]$. Show that there exists $z \in \mathbb{N}$ such that

$$\frac{|(G-z) \cap H|}{n} \geq \frac{|G|}{m} \cdot \frac{|H|}{n} - \frac{|H|}{m}$$

E3. For $n \in \mathbb{N}$, let $E_n \subseteq [n]$. Assume that $\lim_{n\to\infty} |E_n|/n = \gamma > 0$. We will show that there exists a finite set F of cardinality $|F| \leq 1/\gamma$ such that for every m :

$$[m] \subseteq (E_n - E_n) + F, \quad \text{for infinitely many } n. \tag{1}$$

(a) Set $m_0 = 0$, $F = \{m_0\}$ $\Gamma_0 = \mathbb{N}$, $\Lambda_0 = \emptyset$. Assume that $(F_{i-1}, m_{i-1}, \Gamma_{i-1}, \Lambda_{i-1})$ were defined. Construct $(F_i, m_i, \Gamma_i, \Lambda_i)$ following the next process: If for every $m \in \mathbb{N}$ the set

$$\{n \in \Gamma_i : [m] \subseteq (E_n - E_n) + F_i\}$$

is infinite we stop. Otherwise, set m_{i+1} as the minimal $m \in \mathbb{N}$ for which (1) does not hold and set $F_{i+1} = \{m_1, \dots, m_{i+1}\}$,

$$\Gamma_{i+1} = \{n \in \Gamma_i : [m_{i+1} - 1] \subseteq (E_n - E_n) + F_{i+1}\}$$

and

$$\Lambda_{i+1} = \{n \in \mathbb{N} : m_{i+1} \in (E_n - E_n) + F_{i+1}\}.$$

Show that for each $i \in \mathbb{N}$, for which $(F_i, m_i, \Gamma_i, \Lambda_i)$ were defined, if $N \in \Gamma_i \setminus (\Lambda_1 \cup \cdots \cup \Lambda_i)$, then

$$\frac{|E_N|}{N} \leq \frac{1}{i} + \frac{m_i}{iN}.$$

(b) Show that the previous process must stop at step $i \leq 1/\gamma$. Conclude.