Exercise Set Solutions #13

Combinatorial Number Theory (2025)

E1. Show that for any infinite set $E \subseteq \mathbb{N}$, the set of differences $E - E$ is intersective.

> ***Solution:*** Enumerate $E = \{n_1 < n_2 < \dots\}$. Let $A \subseteq \mathbb{N}$ with $\overline{d}(A) = \delta > 0$. Let $N > \delta^{-1}$, and consider the sets $(A - n_1), \dots, (A - n_N)$. Since $N\overline{d}(A) > 1$, there must be $1 \le i < j \le N$ such that $\overline{d}\left((A - n_i) \cap (A - n_j)\right) > 0$. Shifting by n_i, we conclude $\overline{d}\left(A \cap (A - (n_j - n_i))\right) > 0$, and $n_j - n_i \in E - E$.

E2. Show that if $R \subset \mathbb{N}$ is intersective, then for any $A \subset \mathbb{N}$ with $\overline{d}(A) > 0$ there is $n \in R$ such that $\overline{d}(A \cap (A - n)) > 0$.
Hint: Let $\delta = \overline{d}(A)$. Consider $N = N(\frac{\delta^2}{4}) + 1$, where $N(\frac{\delta^2}{4})$ satisfies (ii) of Corollary 84. Then by partitioning the set A into sets A_i each of length N, show that $|A_i| \ge \frac{\delta^2}{4}N$ for "many" $i \in \mathbb{N}$.

> ***Solution:*** Let $A \subset \mathbb{N}$ with $\overline{d}(A) = \delta$. Pick $N = N(\frac{\delta^2}{4}) + 1$ where $N(\frac{\delta^2}{4})$ satisfies (ii) of Corollary 84 in the lecture notes. Define $A_i = A \cap [iN, (i+1)N)$, for $i \in \mathbb{N}_0$, and note that these sets partition the set A.
>
> **Claim:** $|A_i| > \frac{\delta^2}{4}N$ for i in a set of positive density.
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> To prove the claim, since A has upper density δ, then for large N, $|A \cap [1, M]| \ge \frac{\delta}{2}M$, or equivalently, for large k,
>
> $$\frac{1}{k+1}\sum_{0 \le i \le k} |A_i| \ge \frac{\delta^2}{2}N.$$
>
> Suppose for sake of contradiction that $|\{0 \le i \le k \colon |A_i| \ge \frac{\delta^2}{4}N\}| \le \frac{\delta^2}{4}(k+1)$ for large k. Then we have
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> $$\begin{aligned}\frac{\delta^2}{2}N \le \frac{1}{k+1}\sum_{0 \le i \le k}|A_i| &= \frac{1}{k+1}\sum_{0 \le i \le k}|A_i|1_{\{i \colon |A_i| \ge \frac{\delta^2}{4}N\}} + \frac{1}{k+1}\sum_{0 \le i \le k}|A_i|1_{\{i \colon |A_i| < \frac{\delta^2}{4}N\}} \\ &< \frac{\delta^2}{4}N + \frac{\delta^2}{4}N = \frac{\delta^2}{2}N,\end{aligned}$$
>
> yielding a contradiction. This implies the claim.
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> By the choice of N and the claim, for all i in a set of positive density, there are $a_i \in A_i$ and $n_i \in R \cap [1, N]$ such that $\{a_i, a_i + n_i\} \in A$. Using the pigeonhole principle we see that $n_i = n$ for some $n \in R$, for i in a set of positive density. This concludes the proof.

E3. Prove the family of intersective sets is partition regular.

> ***Solution:*** Let us first prove the following claim: if R is intersective, then for every $\delta > 0$, there exists $N(\delta) \in \mathbb{N}$ such that if $N_1, N_2 \ge N(\delta)$ and $A_i \subseteq \{1, \dots, N_i\}$ with $|A_1| \cdot |A_2| \ge \delta N_1 N_2$, then $(A_1 - A_1) \cap (A_2 - A_2) \cap R \ne \emptyset$.
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> Let $C = A_1 \times A_2 \subseteq \{1, \dots, N_1\} \times \{1, \dots, N_2\}$. Then $|C| \ge \delta N_1 N_2$. By symmetry, we may

assume $N_1 \leq N_2$. We can partition C into diagonal slices

$$C = \bigcup_{n=-(N_1-1)}^{N_2-1} (C \cap \{(x, x+n) : x \in \{1, \dots, N_1\}\}) = \bigcup_{n=-(N_1-1)}^{N_2-1} C_n.$$

By the pigeonhole principle, there exists n such that

$$|C_n| \geq \delta \frac{N_1 N_2}{N_1 + N_2 - 1} > \frac{\delta}{2} N_1.$$

Write $C_n = \{(x_1, x_1+n), \dots, (x_k, x_k+n)\}$ with $1 \leq x_1 < \cdots < x_k \leq N_1$ and $k > \frac{\delta}{2} N_1$. By the compactness principle for intersective sets, if N_1 is sufficiently large, there exists $r \in R$ and $1 \leq i < j \leq k$ such that $x_j - x_i = r$. Since $C_n \subseteq A_1 \times A_2$, it follows that $x_i, x_j \in A_1$ and $(x_i + n), (x_j + n) \in A_2$. Hence, $r \in (A_1 - A_1) \cap (A_2 - A_2)$.

Now suppose $R \subseteq \mathbb{N}$ is an intersective set and $R = R_1 \cup R_2$. Assume R_1 is not an intersective set. Then there exists $A \subseteq \mathbb{N}$ such that $\overline{d}(A) = \alpha > 0$ and $R_1 \cap (A - A) = \emptyset$. Hence, there is a sequence $N_n \to \infty$ such that $A_n = A \cap \{1, \dots, N_n\}$ satisfies $|A_n| > \frac{\alpha}{2} N_n$ and $R_1 \cap (A_n - A_n) = \emptyset$ for every $n \in \mathbb{N}$. Let $B \subseteq \mathbb{N}$ with $\overline{d}(B) = \beta > 0$, and let $M_n \to \infty$ such that $B_n = B \cap \{1, \dots, M_n\}$ satisfies $|B_n| > \frac{\beta}{2} M_n$ for $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$ such that $\min\{M_n, N_n\} \geq N\left(\frac{\alpha\beta}{4}\right)$, the claim above implies that $R \cap (A_n - A_n) \cap (B_n - B_n) \neq \emptyset$. In particular, $R_2 \cap (B_n - B_n) \neq \emptyset$. Since B is an arbitrary set of positive upper density, this implies that R_2 is an intersective set.