Exercise Set Solutions #12
Combinatorial Number Theory (2025)

Let $\|x\| = \min\{|x-n| : n \in \mathbb{Z}\}$ denote the distance of x to the closest integer.

E1. Let R be an intersective set. Show that for every $\alpha \in \mathbb{R}$ and every $\varepsilon > 0$, there exists $r \in R$ such that $\|r\alpha\| < \varepsilon$.

> ***Solution:*** Let $A = \{n \in \mathbb{N} : \|n\alpha\| < \frac{\varepsilon}{2}\}$. For $n, m \in A$, $\|(n-m)\alpha\| \leq \|n\alpha\| + \|m\alpha\| < \varepsilon$, so $A - A \subseteq \{n \in \mathbb{N} : \|n\alpha\| < \varepsilon\}$. Moreover, by the pigeonhole principle, A is syndetic with gaps bounded by $\frac{2}{\varepsilon}$, so $d(A) > 0$. Therefore,
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> $$R \cap \{n \in \mathbb{N} : \|n\alpha\| < \varepsilon\} \supseteq R \cap (A - A) \neq \emptyset.$$

E2. Suppose R is intersective. Show that the following sets are also intersective:

- **(a)** $aR = \{ar : r \in R\}$ for any $a \in \mathbb{N}$.
- **(b)** $R/a = \{n \in \mathbb{N} : an \in R\}$ for any $a \in \mathbb{N}$.
- **(c)** $R \setminus F$ for any finite subset $F \subseteq R$.

> ***Solution:***
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> **(a)** Let $A \subseteq \mathbb{N}$ be such that $\overline{d}(A) > 0$, and write $A = A_0 \cup A_1 \cup \cdots \cup A_{a-1}$ with $A_i = \{n \in A : n \equiv i \pmod a\}$. Clearly the sets are disjoint, and by sub-additivity of upper density we know that at least one A_i is such that $\overline{d}(A_i) > 0$. By construction, the set $A_i - A_i$ only contains multiples of a, so we may consider the set $A' = \frac{1}{a}A_i$. The set A' has upper density $\overline{d}(A') = a \cdot \overline{d}(A_i) > 0$, so there exist $x, y \in A'$ such that $y - x \in R$, since R is intersective. Write $x = \frac{x'}{a}$ and $y = \frac{y'}{a}$ with $x', y' \in A_i$. Then $x' = ax$ and $y' = ay$, so $y' - x' = a(y-x) \in aR$. Hence, $aR \cap (A - A) \supseteq aR \cap (A_i - A_i) \neq \emptyset$, so aR is intersective.
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> **(b)** Let $A \subseteq \mathbb{N}$ be such that $\overline{d}(A) > 0$ and consider the set $A' = aA$, with upper density given by $\overline{d}(A') = \overline{d}(aA) = \frac{1}{a}\overline{d}(A) > 0$. By the intersective property of R, there exist $x, y \in A'$ such that $y - x \in R$. Writing $x = ax'$ and $y = ay'$ for some $x', y' \in A$, we have that $x' - y' \in R/a$, so $R/a \cap (A - A) \neq \emptyset$. That is, R/a is intersective.
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> **(c)** Given a set $A \subseteq \mathbb{N}$ with $\overline{d}(A) > 0$, our goal is to find a subset $A' \subseteq A$ such that $\overline{d}(A') > 0$ and $F \cap (A' - A') = \emptyset$. Partition $A = A_0 \cup A_1 \cup \cdots \cup A_k$ with $A_i = \{n \in A : n \equiv i \pmod{k+1}\}$, where $k = \max F$. Then $A_i - A_i \subseteq (k+1)\mathbb{N}$, so in particular, $F \cap (A_i - A_i) = \emptyset$. As in part (i), we may choose $i \in \{0, 1, \ldots, k\}$ such that $\overline{d}(A_i) > 0$. Since R is intersective, $R \cap (A_i - A_i) \neq \emptyset$. By the observation that $F \cap (A_i - A_i) = \emptyset$, we conclude that
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> $$(R \setminus F) \cap (A - A) \supseteq R \cap (A_i - A_i) \neq \emptyset.$$
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> Therefore, $R \setminus F$ is intersective.

E3. Show that if $R \subseteq \mathbb{N}$ is intersective, then there are disjoint intersective sets $T, S \subseteq \mathbb{N}$ such that $R = S \cup T$.

Hint: Show that we can partition the set R into a union of infinitely many pairwise disjoint finite sets R_k such that for any $k \in \mathbb{N}$ and any $E \subseteq \mathbb{N}$ with $\overline{d}(E) \geq 1/k$, there is $n \in R_k$ such that $E \cap (E - n) \neq \emptyset$.

> ***Solution:*** According to the hint, first we partition R into infinitely many pairwise disjoint finite sets R_k such that for any $k \in \mathbb{N}$ and any $E \subset \mathbb{N}$ with $\overline{d}(E) > \frac{1}{k}$, there is $n \in R_k$ with $E \cap (E - n) \neq \emptyset$.
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> We take $\delta_1 = 1$ and since R is intersective, there is $N_1 \in \mathbb{N}$ such that for all $E \subset \mathbb{N}$ with $\overline{d}(E) \geq \delta_1$ there is $n \in R \cap \{1, \ldots, N_1\}$ with $E \cap (E - n) \neq \emptyset$. Now we set $R_1 = R \cap \{1, \ldots, N_1\}$.
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> Suppose we have already defined $R_1, \ldots, R_k$. We want to define R_{k+1}. Since $R_1, \ldots, R_k$ are all finite, their union is finite, and then $R' = R \setminus (R_1 \cup \ldots \cup R_k)$ is intersective. We take $\delta_{k+1} = \frac{1}{k+1}$ and then there is $N_{k+1} \in \mathbb{N}$ such that for all $E \subset \mathbb{N}$ with $\overline{d}(E) \geq \delta_{k+1}$ there is $n \in R' \cap \{1, \ldots, N_{k+1}\}$ with $E \cap (E - n) \neq \emptyset$. We set $R_{k+1} = R' \cap \{1, \ldots, N_{k+1}\}$.
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> Now we define
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> $$S = \bigcup_{i \in \mathbb{N}} R_{2i} \quad \text{and} \quad T = \bigcup_{i \in \mathbb{N}} R_{2i-1}.$$
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> By construction, we have that $R = S \cup T$ disjoint union. It remains to show that S, T are both intersective.
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> Take any $E \subset \mathbb{N}$ with $\overline{d}(E) > 0$. Then there exists $k \in \mathbb{N}$ such that $\overline{d}(E) \geq \frac{1}{k} \geq \frac{1}{2k-1} \geq \frac{1}{2k}$. Then by construction, there is $n \in R_{2k} \subset S$ and $m \in R_{2k-1} \subset T$ such that $E \cap (E - n) \neq \emptyset$ and $E \cap (E - m) \neq \emptyset$. This concludes the proof.