Exercise Set #12
Combinatorial Number Theory (2025)

Let $\|x\| = \min\{|x-n| : n \in \mathbb{Z}\}$ denote the distance of x to the closest integer.

E1. Let R be an intersective set. Show that for every $\alpha \in \mathbb{R}$ and every $\varepsilon > 0$, there exists $r \in R$ such that $\|r\alpha\| < \varepsilon$.

E2. Suppose R is intersective. Show that the following sets are also intersective:

- **(a)** $aR = \{ar : r \in R\}$ for any $a \in \mathbb{N}$.
- **(b)** $R/a = \{n \in \mathbb{N} : an \in R\}$ for any $a \in \mathbb{N}$.
- **(c)** $R \setminus F$ for any finite subset $F \subseteq R$.

E3. Show that if $R \subseteq \mathbb{N}$ is intersective, then there are disjoint intersective sets $T, S \subseteq \mathbb{N}$ such that $R = S \cup T$.
Hint: Show that we can partition the set R into a union of infinitely many pairwise disjoint finite sets R_k such that for any $k \in \mathbb{N}$ and any $E \subseteq \mathbb{N}$ with $\overline{d}(E) \geq 1/k$, there is $n \in R_k$ such that $E \cap (E - n) \neq \emptyset$.