

## Exercise Set Solutions #11

### Combinatorial Number Theory (2025)

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- E1.** Let  $A \subseteq \mathbb{Z}_N$  with  $|A| > \frac{2}{3}N$ . Prove  $A$  contains a nontrivial 3-term arithmetic progression (without using Roth's theorem). **Hint:** Consider the set  $A \cap (A - d) \cap (A - 2d)$ .

**Solution:** Let  $d \in \mathbb{N}$ . Then

$$|A \cap (A - d) \cap (A - 2d)| = N - |A^c \cup (A - d)^c \cup (A - 2d)^c| > N - \left( \frac{1}{3}N + \frac{1}{3}N + \frac{1}{3}N \right) = 0.$$

In particular,  $A \cap (A - d) \cap (A - 2d) \neq \emptyset$ . Let  $a \in A \cap (A - d) \cap (A - 2d)$ . This means  $a \in A$ ,  $a + d \in A$ , and  $a + 2d \in A$ , so  $A$  contains the 3-term arithmetic progression  $\{a, a + d, a + 2d\}$ .

- E2.** Show that for any  $0 < \delta < 1$ , there exists  $N_0 = N_0(\delta)$  such that for any  $N \geq N_0$  there exists a set  $A \subset \{1, \dots, N\}$  with  $|A| \geq \frac{\delta}{2}N$  contains  $\ll \delta^{C \log(1/\delta)} N^2$  3-term arithmetic progressions, for some absolute constant  $C > 0$ .

**Hint:** Use Behrend's theorem to find a set  $A_0 \subset \{1, \dots, M\}$  with  $M\delta$  elements that is free of 3-term arithmetic progressions, for a suitable  $M = M(C, \delta)$ .

**Solution:** Behrend's theorem asserts that for sufficiently large  $M$ , there is a set  $A_0 \subset \{1, \dots, M\}$  with  $|A_0| \geq M \exp(-c\sqrt{\log M})$  that is free of 3-term arithmetic progressions, where  $c > 0$  is an absolute constant. Let  $C = c^{-2} > 0$  and choose  $M \gg \exp(C(\log \delta)^2)$ . Then by Behrend's theorem, there is a set  $A_0 \subset \{1, \dots, M\}$  with  $M\delta$  elements. Let  $N = 6M$ . Consider the set  $A \subset \{1, \dots, N\}$  consisting of all  $a \leq N$  with  $a \equiv a_0 \pmod{2M}$  for some  $a_0 \in A_0$ . Then  $|A| = 3M\delta = \frac{\delta}{2}N$ . Now notice that if  $x, y, z \in A$  form a 3-term arithmetic progression, then  $x \equiv y \equiv z \pmod{2M}$ , as  $A_0$  does not contain nontrivial arithmetic progressions. This implies that the number of 3-term arithmetic progressions in  $A$  is exactly

$$|A_0| = \delta M \ll (\delta/M)N^2 \ll \delta^{C \log(1/\delta)} N^2.$$

- E3.** Show that for every irrational  $\alpha$ , there are infinitely many coprime  $p \in \mathbb{Z}$  and  $q \in \mathbb{N}$  with

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}.$$

**Hint:** Use Dirichlet's approximation theorem.

**Solution:** By Dirichlet's approximation theorem, we have that for any irrational number  $\alpha$  and any  $Q \in \mathbb{N}$  there exists  $p, q \in \mathbb{Z}$ , with  $1 \leq q \leq Q$  such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{qQ} \leq \frac{1}{q^2}.$$

In particular, there exist integer sequences  $(p_n)_{n \in \mathbb{N}}$  and  $(q_n)_{n \in \mathbb{N}}$  with  $1 \leq q_n \leq n$  such that

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n n} \leq \frac{1}{q_n^2}$$

for every  $n \in \mathbb{N}$ . Dividing  $p_n$  and  $q_n$  by  $\gcd(p_n, q_n)$ , we may assume that  $p_n$  and  $q_n$  are coprime for every  $n \in \mathbb{N}$ . We claim that  $q_n \rightarrow \infty$  so that the sequence  $\frac{p_n}{q_n}$  consists of infinitely many different rational numbers. To prove this we proceed by contradiction. Note that

$$|q_n \alpha - p_n| < \frac{1}{n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Suppose that  $q_n$  is bounded. Then, since  $|q_n \alpha - p_n| < \frac{1}{n}$ , we have  $q_n \alpha - \frac{1}{n} < p_n < q_n \alpha + \frac{1}{n}$ , so  $p_n$  is also bounded. Therefore,  $|q_n \alpha - p_n|$  can take only finitely many different values, but  $\alpha$  is irrational and so we get a contradiction to the fact that  $|q_n \alpha - p_n| \rightarrow 0$  since  $q_n \alpha - p_n \neq 0$ . Thus, there exist infinitely many coprime  $p \in \mathbb{Z}$  and  $q \in \mathbb{N}$  with

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}.$$