Exercise Set #11
Combinatorial Number Theory (2025)

E1. Let $A \subseteq \mathbb{Z}_N$ with $|A| > \frac{2}{3}N$. Prove A contains a nontrivial 3-term arithmetic progression (without using Roth's theorem).
Hint: Consider the set $A \cap (A-d) \cap (A-2d)$.

E2. Show that for any $0 < \delta < 1$, there exists $N_0 = N_0(\delta)$ such that for any $N \geq N_0$ there exists a set $A \subset \{1, \ldots, N\}$ with $|A| \geq \frac{\delta}{2}N$ contains $\ll \delta^{C \log(1/\delta)} N^2$ 3-term arithmetic progressions, for some absolute constant $C > 0$.
Hint: Use Behrend's theorem to find a set $A_0 \subset \{1, \ldots, M\}$ with $M\delta$ elements that is free of 3-term arithmetic progressions, for a suitable $M = M(C, \delta)$.

E3. Show that for every irrational α, there are infinitely many coprime $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ with

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Hint: Use Dirichlet's approximation theorem.