MATH-329 Nonlinear optimization
Exercise session 9: Constraint qualification

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1. Constraint qualification or not? For the following sets defined by equality and inequality constraints, determine whether they satisfy constraint qualifications everywhere or if they fail at some point.

- $S = \{x \in \mathbb{R}^n \mid Ax = b \quad \text{and} \quad Cx \leq d\}$ for some $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, C \in \mathbb{R}^{p\times n}, d \in \mathbb{R}^p$.
- $S = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$. More generally, what can you say in general about a set S defined as the graph of a differentiable function?
- $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 = y^3\}$.
- $S = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^3 + x^2\}$.
- $S = \{(x, y) \in \mathbb{R}^2 \mid (x - 1/2)^2 + y^2 \leq 1 \quad \text{and} \quad (x + 1/2)^2 + y^2 \leq 1\}$.

Answer. If it holds we use LICQ or another theorem from the course. If it does not hold we compute F_xS and T_xS, and we show they are not the same.

- Let $x \in S$. We showed in the course that if the active constraint at x are affine then $\mathrm{F}_xS = \mathrm{T}_xS$ (Theorem 8.15). This condition clearly holds since S is described by affine equalities and inequalities.
- Constraint qualifications hold. We show a more general result. Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable and let $S = \{(x, y) \in \mathbb{R}^{n+1} \mid y = f(x)\}$ be the graph of f. Then constraint qualifications hold everywhere on S. Indeed, S is described by a single equality constraint, $h(x, y) = 0$ where $h(x, y) = y - f(x)$. We compute that the gradient of h is

  $$\nabla h(x, y) = \begin{bmatrix} \nabla f(x) \\ 1 \end{bmatrix}.$$

  The gradient at (x, y) is non-zero so constraint qualifications are satisfied.
- The set is described by a single equality constraint, $h(x, y) = 0$, where $h(x, y) = x^2 - y^3$ (see Figure 1). We compute that for all x, y we have

  $$\nabla h(x, y) = \begin{bmatrix} 2x \\ -3y^2. \end{bmatrix}$$

  In particular the gradient is zero at the origin: it means that $\mathrm{F}_{(0,0)}S = \mathbb{R}^2$. Clearly, the tangent cone at $(0, 0)$ is not $\mathbb{R}^2$ so constraint qualifications do not hold at this point.

  For all other $(x, y) \in S$ the gradient $\nabla h(x, y)$ is non-zero so constraint qualifications hold.

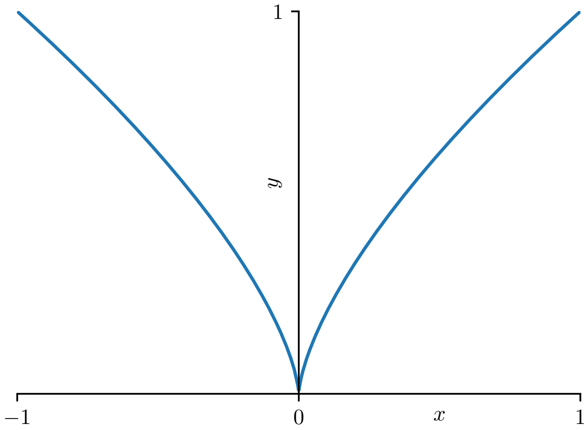

Figure 1: $x^2 = y^3$.

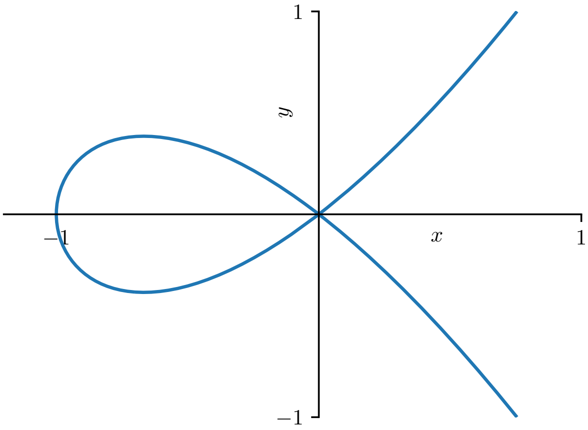

Figure 2: $y^2 = x^3 + x^2$.

- The set is described by a single equality constraint, $h(x, y) = 0$, where $h(x, y) = y^2 - x^3 - x^2$ (see Figure 2). We compute that for all x, y we have

  $$\nabla h(x, y) = \begin{bmatrix} -3x^2 - 2x \\ 2y. \end{bmatrix}$$

  Here again, at the origin constraint qualifications do not hold because $\mathrm{F}_{(0,0)}S = \mathbb{R}^2$ but $\mathrm{T}_{(0,0)}S \neq \mathbb{R}^2$.

  For all $(x, y) \in S$ that is non-zero the gradient is non-zero so constraint qualifications hold.

- The set is described by two inequality constraints, $g_1(x, y) \leq 0$ and $g_2(x, y) \leq 0$ where $g_1(x, y) = (x - 1/2)^2 + y^2 - 1$ and $g_2(x, y) = (x + 1/2)^2 + y^2 - 1$. It is the intersection of two disks (see Figure 3). Constraint qualifications hold at all points because Slater's condition holds. Indeed, we have $g_1(0, 0) < 0$ and $g_2(0, 0) < 0$. So the point $(0, 0)$ satisfies the inequality constraints strictly.

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2. A particular stationary point. Let $\mathcal{E} = \mathbb{R}^2$. Consider the function $f(x) = x_2$, to be minimized on the set $S = \{x \in \mathbb{R}^2 : \|x\| \geq 1\}$ (the complement of the open unit disk). Consider

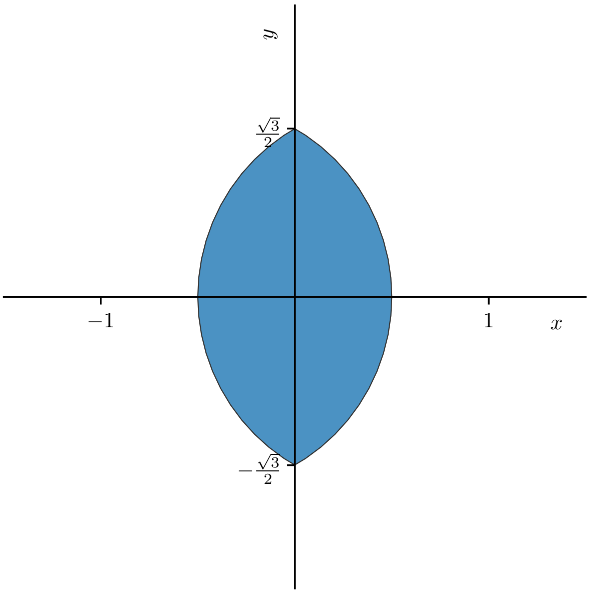

Figure 3: Intersection of two disks.

the special point $x^* = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$.

1. Draw the situation.
2. Show that x^* is stationary for f on S (you can do this using normal cones for example).
3. Show that x^* is not a local minimum for f on S.
4. What does this exercise highlight about the optimality conditions discussed in class?

Answer.

1. See Figure 4.

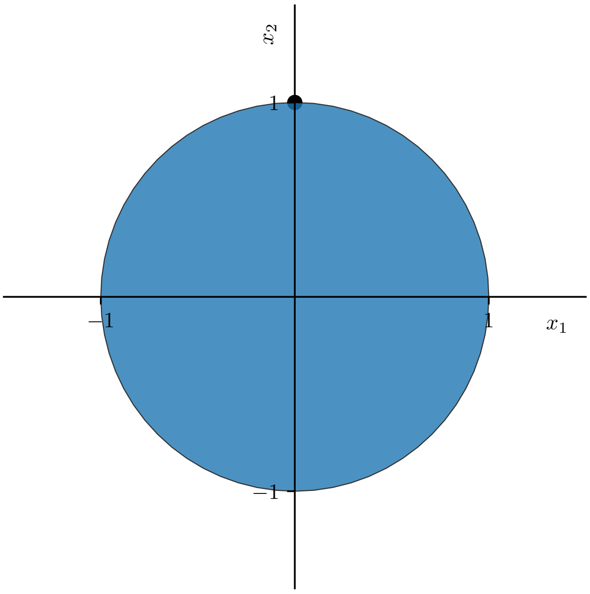

Figure 4: Unit disk.

2. We show that x^* is a stationary point by showing that
$$-\nabla f(x^*) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \in (T_{x^*}S)^\circ.$$
Here we have that $T_{x^*}S = \left\{(v_1, v_2)^\top : v_1 \in \mathbb{R}, v_2 \geq 0\right\}$ so that $(T_{x^*}S)^\circ = \left\{(0, w)^\top : w \leq 0\right\}$.
3. We let
$$c(t) = (\sin(t), \cos(t))$$
be a curve on S such that $c(0) = x^*$. Then we find that $f(c(t)) < f(c(0))$ for t sufficiently small. This shows that x^* is not a local minimum, but rather a saddle-point (as it is first-order stationary).
4. This highlights that, similarly to the unconstrained case, stationary conditions are indeed only *necessary* conditions for optimality. In general, they are not sufficient (as we see here).

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3. Intersection of disks. We let $a \geq 0$ be a real parameter and define the set $S \subseteq \mathbb{R}^2$ through the two following inequality constraints: $g_1(x) = (x_1 - a)^2 + x_2^2 - 1 \leq 0$ and $g_2(x) = (x_1 + a)^2 + x_2^2 - 1 \leq 0$.

1. Draw the set S for a few interesting values of a. (Which values? Think about it.)
2. Consider the point $x = (0, \sqrt{1 - a^2})$; show it in your drawings. For which values of a is x in S?
3. For which values of a does the LICQ constraint qualification hold at x, and for which does it not?
4. Same question for MFCQ.
5. What is your conclusion regarding the relationship between the tangent cone and the cone of linearized feasible directions at x? Discuss carefully as a function of a. If there is a value of a for which none of the CQ holds, figure out the tangent cone "by hand".

Answer.

1. The set S is the intersection of the two closed unit disks centered at $(-a, 0)$ and $(a, 0)$. This intersection is non-empty when $a \leq 1$. In the edge-case where $a = 1$ the set S contains the unique point $(0, 0)$. The other edge case is $a = 0$ where the two disks are identical. It's also interesting to consider a value of a strictly between 0 and 1.
2. We find that $g_1(x) = g_2(x) = 0$ whenever $\sqrt{1 - a^2}$ is well-defined (that is, when $a \leq 1$). So the point x is always in S with $a \in [0, 1]$.
3. We find that
$$\nabla g_1(x) = \begin{bmatrix} 2(x_1 - a) \\ 2x_2 \end{bmatrix} = \begin{bmatrix} -2a \\ 2\sqrt{1 - a^2} \end{bmatrix} \qquad \nabla g_2(x) = \begin{bmatrix} 2(x_1 + a) \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 2a \\ 2\sqrt{1 - a^2}. \end{bmatrix}$$
Both constraints are active and the gradients are linearly dependent if and only if $a = 0$ or $a = 1$. So the LICQ constraint qualification holds for all $0 < a < 1$.

4. We know that MFCQ holds for all $a \in (0,1)$ because LICQ implies MFCQ. When $a = 0$ the gradients are given by

$$\nabla g_1(x) = \nabla g_2(x) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

and $x = (0,1)^\top$. If we take $\bar{x} = (0,0)^\top$ then

$$\begin{aligned} \langle g_i(x), \bar{x} - x \rangle &= \big\langle (0,2)^\top, (0,-1) \big\rangle \\ &< 0, \end{aligned}$$

so MFCQ holds in this case.

When $a = 1$ the gradients are given by

$$\nabla g_1(x) = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \qquad \nabla g_2(x) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

and $x = (0,0)^\top$. We see here that MFCQ cannot hold. Indeed, if they did, then we would have equality of the tangent cone and of the linearized cone, but they are different.

5. In fine we find that $\mathrm{F}_x S = \mathrm{T}_x S$ when $a \in [0,1[$. When $a = 1$ we have $\mathrm{T}_x S \subsetneq \mathrm{F}_x S$. When $a > 1$ the feasible set is empty so the question is meaningless.

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Supplementary exercises

The following two exercises establish that the dual and the polar of a cone are *always* closed, convex cones.

1. Dual and polar cones are closed. Let C be a cone. Show that C^* and C° are closed cones (even if C is not closed.) *Hint*: recall what happens when we take the intersection of infinitely many closed sets.

Answer. Since $C^* = -C^\circ$, it is sufficient to argue for C°. Polars are clearly cones: if $\langle w, v \rangle \leq 0$ for all $v \in C$, then $\langle \alpha w, v \rangle \leq 0$ for all $v \in C$ and all $\alpha > 0$. To see that the polar is closed, recall that the intersection of an arbitrary collection of closed sets is closed, then notice that we can write C° as the intersection of a collection of (closed) half spaces:

$$C^\circ = \bigcap_{v \in C} \{ w \in \mathcal{E} : \langle w, v \rangle \leq 0 \}.$$

(On the other hand, a *union* of closed sets may fail to be closed if there are infinitely many sets we are taking the union of.) ■

2. Dual and polar cones are always convex sets. Remember that a set S is convex if for all $x, y \in S$ and all $t \in [0,1]$ we have $(1-t)x + ty \in S$. Show that the dual and the polar of a cone C are convex (even when C is not convex).

Answer. Let $C \subseteq \mathcal{E}$ be a cone. Then the dual cone is given by

$$C^* = \{w \in \mathcal{E} \mid \langle w, v \rangle \geq 0 \text{ for all } v \in C\}.$$

Let $w_1, w_2 \in C^*$ and $t \in [0, 1]$. Then for all $v \in C$ we have

$$\begin{aligned} \langle (1-t)w_1 + tw_2, v \rangle &= (1-t) \langle w_1, v \rangle + t \langle w_2, v \rangle \\ &\geq 0, \end{aligned}$$

which shows that $(1-t)w_1 + tw_2 \in C^*$. We conclude that C^* is convex. This automatically implies that C° is convex (or you can rewrite the argument with the reversed inequality if you want to be explicit). ■

3. Polar and dual invert inclusion. Let C and C' be two cones in $\mathcal{E}$ such that $C \subseteq C'$.

1. Show that $(C')^\circ \subseteq C^\circ$.
2. Show that $(C')^* \subseteq C^*$

Answer.

1. Let $w \in (C')^\circ$. This means that for all $v \in C'$, we have $\langle w, v \rangle \leq 0$. Now, take $v \in C$. By the inclusion, we know that $v \in C'$ as well. Therefore we have $\langle w, v \rangle \leq 0$, which implies $w \in C^\circ$. Repeating for all $w \in (C')^\circ$ shows $(C')^\circ \subseteq C^\circ$.
2. Since we have $C^* = -C^\circ = \{-v : v \in C^\circ\}$, the result is immediate.

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