# MATH-329 Nonlinear optimization
# Exercise session 11: Lagrangian Duality

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**1. Minimization of quadratic function on affine subspace.** Let $H \in \mathbb{R}^{n\times n}, A \in \mathbb{R}^{p\times n}, b \in \mathbb{R}^p, c \in \mathbb{R}^n$ and assume that $H$ is positive definite. Consider the optimization problem

$$\min f(x) \quad \text{subject to} \quad x \in S,$$

where $f(x) = \frac{1}{2}x^\top Hx + c^\top x$ and $S = \{x \in \mathbb{R}^n \mid Ax = b\}$. We assume $S$ is non-empty.

1. Write the Lagrangian function $L$ for this problem. What is its domain?
2. What is the primal function $L_P$ and its domain?
3. Obtain an explicit expression for the dual function $L_D$. What is its domain? What is the dual problem?
4. Find a characterization for points that solve the dual problem. Hint: You should find that they are defined by a linear system of equations.
5. Argue that strong duality holds by checking the assumptions of the strong duality theorem explicitly. Use your reasoning here to argue that the linear system of equations you found in the previous question has a solution.
6. Use strong duality to solve the primal problem. Check that the solution satisfies the constraints.

As a side note for context: the exercise would become quite a bit harder (and interesting) if we only assume $H$ positive *semi*definite.

**Answer.**

1. The Lagrangian $L\colon \mathbb{R}^n \times \mathbb{R}^p$ is given by

$$L(x,\mu) = \frac{1}{2}x^\top Hx + c^\top x + \mu^\top(Ax - b).$$

2. As always, the primal function is

$$L_P(x) = \sup_{\mu\in\mathbb{R}^p} L(x,\mu) = \begin{cases} f(x) & \text{if } x \in S \\ +\infty & \text{otherwise.} \end{cases}$$

3. The dual function is given by

$$\begin{aligned} L_D(\mu) &= \inf_{x \in \mathbb{R}^n} L(x, \mu) \\ &= \inf_{x \in \mathbb{R}^n} \frac{1}{2} x^\top H x + (c^\top + \mu^\top A)x - \mu^\top b. \end{aligned}$$

The infimum of $x \mapsto \frac{1}{2}x^\top Hx + (c^\top + \mu^\top A)x - \mu^\top b$ is attained at $x = -H^{-1}(A^\top \mu + c)$ using the fact that the function is strongly convex and that its unique minimum is attained when the gradient is zero. So we find that

$$\begin{aligned} L_D(\mu) &= \frac{1}{2}(\mu^\top A + c^\top)H^{-1}(A^\top \mu + c) - (\mu^\top A + c^\top)H^{-1}(A^\top \mu + c) - \mu^\top b \\ &= -\frac{1}{2}\mu^\top A H^{-1} A^\top \mu - (\mu^\top A + c^\top)H^{-1}c - \mu^\top b \\ &= -\frac{1}{2}\mu^\top A H^{-1} A^\top \mu - (AH^{-1}c + b)^\top \mu - c^\top H^{-1} c. \end{aligned}$$

The dual problem consists in maximizing $L_D$ for $\mu \in \mathbb{R}^p$.

4. The dual function is concave and we want to maximize it. Optimal points are found where the gradient is zero. They satisfy

$$AH^{-1}A^\top \mu + (AH^{-1}c + b) = 0.$$

5. The primal admits a global minimizer as the function is strongly convex on a non-empty closed and convex set (an affine subspace). Constraints are affine so constraint qualifications hold everywhere. Moreover $x \mapsto L(x, \mu)$ is convex for all $\mu$. We conclude that hypotheses from the strong duality theorem hold: we have strong duality.

   Let $x^*$ be any global minimizer for the primal. We know it is a KKT point, and so there exists a vector of Lagrange multipliers $\mu^*$ for $x^*$. Strong duality tells us that $\mu^*$ is optimal for the dual. Therefore, the linear system we have found in the previous question admits $\mu^*$ as a solution: there exists at least one solution.

6. Let $\mu^*$ be a solution of the dual problem, that is, it satisfies

$$AH^{-1}A^\top \mu^* + (AH^{-1}c + b) = 0.$$

   (Existence of $\mu^*$ was argued in the previous question.) The strong duality theorem tells us that a solution $x^*$ to the original problem is a minimum of

$$\min_{x \in \mathbb{R}^n} L(x, \mu^*) = \frac{1}{2} x^\top H x + c^\top x + (\mu^*)^\top (Ax - b).$$

   This is a strongly convex quadratic problem and the gradient is given by $Hx + c + A^\top \mu^*$ for all $x \in \mathbb{R}^n$. This gives that

$$x^* = -H^{-1}(A^\top \mu^* + c).$$

   Thus, to solve the problem, we can solve the linear system that provides us $\mu^*$, then solve the linear system that provides us $x^*$. It is not difficult to check that $Ax^* = b$, that is, $x^*$ satisfies the constraint (as it should).

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