MATH-329 Nonlinear optimization
Exercise session 9: Constraint qualification

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1. Constraint qualification or not? For the following sets defined by equality and inequality constraints, determine whether they satisfy constraint qualifications everywhere or if they fail at some point.

- $S = \{x \in \mathbb{R}^n \mid Ax = b \quad \text{and} \quad Cx \leq d\}$ for some $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, C \in \mathbb{R}^{p\times n}, d \in \mathbb{R}^p$.
- $S = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$. More generally, what can you say in general about a set S defined as the graph of a differentiable function?
- $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 = y^3\}$.
- $S = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^3 + x^2\}$.
- $S = \{(x, y) \in \mathbb{R}^2 \mid (x - 1/2)^2 + y^2 \leq 1 \quad \text{and} \quad (x + 1/2)^2 + y^2 \leq 1\}$.

2. A particular stationary point. Let $\mathcal{E} = \mathbb{R}^2$. Consider the function $f(x) = x_2$, to be minimized on the set $S = \{x \in \mathbb{R}^2 : \|x\| \geq 1\}$ (the complement of the open unit disk). Consider the special point $x^* = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$.

1. Draw the situation.
2. Show that x^* is stationary for f on S (you can do this using normal cones for example).
3. Show that x^* is not a local minimum for f on S.
4. What does this exercise highlight about the optimality conditions discussed in class?

3. Intersection of disks. We let $a \geq 0$ be a real parameter and define the set $S \subseteq \mathbb{R}^2$ through the two following inequality constraints: $g_1(x) = (x_1 - a)^2 + x_2^2 - 1 \leq 0$ and $g_2(x) = (x_1 + a)^2 + x_2^2 - 1 \leq 0$.

1. Draw the set S for a few interesting values of a. (Which values? Think about it.)
2. Consider the point $x = (0, \sqrt{1 - a^2})$; show it in your drawings. For which values of a is x in S?
3. For which values of a does the LICQ constraint qualification hold at x, and for which does it not?
4. Same question for MFCQ.
5. What is your conclusion regarding the relationship between the tangent cone and the cone of linearized feasible directions at x? Discuss carefully as a function of a. If there is a value of a for which none of the CQ holds, figure out the tangent cone "by hand".

Supplementary exercises

The following two exercises establish that the dual and the polar of a cone are *always* closed, convex cones.

1. Dual and polar cones are closed. Let C be a cone. Show that C^* and C° are closed cones (even if C is not closed.) *Hint*: recall what happens when we take the intersection of infinitely many closed sets.

2. Dual and polar cones are always convex sets. Remember that a set S is convex if for all $x, y \in S$ and all $t \in [0, 1]$ we have $(1-t)x + ty \in S$. Show that the dual and the polar of a cone C are convex (even when C is not convex).

3. Polar and dual invert inclusion. Let C and C' be two cones in $\mathcal{E}$ such that $C \subseteq C'$.

1. Show that $(C')^\circ \subseteq C^\circ$.
2. Show that $(C')^* \subseteq C^*$