MATH 327 - TOPICS IN COMPLEX ANALYSIS
PRACTICE EXAM SOLUTIONS
TIME: 2 HOURS

Problem 1. True or false ($2 + 2 + 2 + 2 + 2$ points)

For each of the following statements, decide whether it is true or false. Justify each statement you deem true with a short proof. For each claim you deem false, give an explicit counterexample. Please read the claims carefully to not miss a word!

a. Let $D \subset \mathbf{C}$ be a domain. Let $(f_n)_{n\in\mathbf{N}}$ be a sequence of nowhere vanishing holomorphic functions $f_n \colon D \to \mathbf{C}$ converging locally uniformly to a nonconstant function $f \colon D \to \mathbf{C}$. Then $(1/f_n)_{n\in\mathbf{N}}$ converges locally uniformly to $1/f$ on D.

b. Let $S \subset \mathbf{C}$ be a countable set. Then there is an entire function $f \colon \mathbf{C} \to \mathbf{C}$ whose set of zeros is exactly S.

c. Let $f \colon \mathbf{C} \to \mathbf{C}$ be a nonconstant entire function with $f(0) = 1$ and $f'(0) = 0$. Then f has a zero.

d. Let $f \colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be a nonconstant holomorphic function. Assume $f\big|_{\mathbf{C}}$ is not a polynomial. Then f has a pole or an essential singularity.

e. Assume the function $f \colon \mathbf{C}^n \to \mathbf{C}$ is not holomorphic while $g \colon \mathbf{C}^n \to \mathbf{C}^n$ is, where n is at least 2. Then the composition $f \circ g$ is not holomorphic.

Solutions. a. *True* (**0.5 points**). Indeed, from the lecture we know the number of zeros of the members of the sequence cannot increase unless the limit vanishes identically, which is ruled out by the nonconstancy of f. This shows $1/f$ is well-defined (**0.5 points** for the proof of well-definedness). Moreover, from the lecture we know f is holomorphic and thus continuous. These two observations yield that given any compact set $K \subset D$, there exists $R > 0$ such that $|f|\big|_K \subset [2R, \infty)$ (**0.5 points**). By locally uniform convergence of $(|f_n|)_{n\in\mathbf{N}}$ to $|f|$ (a consequence of the triangle inequality), up to removing finitely many members of the sequence $(f_n)_{n\in\mathbf{N}}$ we may and will assume $|f_n|\big|_K \subset [R, \infty)$ for every $n \in \mathbf{N}$. Noting the reciprocal assignment $1/z$ is Lipschitz continuous on the complement of $B_R(0)$, say with Lipschitz constant C_R, the claim easily follows from the following simple inequality (**0.5 points**):

$$\lim_{n\to\infty} \sup \left| \frac{1}{f_n} - \frac{1}{f} \right| (K) \leq C_R \lim_{n\to\infty} \sup |f_n - f|(K) = 0.$$

b. *False* (**0.5 points**). Consider the set $S := \{1/n : n \in \mathbf{N}\}$ (**0.5 points** for an appropriate example). It has an accumulation point at zero. If an entire function $f \colon \mathbf{C} \to \mathbf{C}$ vanishes on S, the identity theorem forces f to vanish identically. This is in conflict with the requirement of f to be nonvanishing outside S (**1 point** for the correct proof that the example falsifies the claim).

c. *False* (**0.5 points**). For instance, the assignment $f(z) := \mathrm{e}^{z^2}$ (**0.5 points** for an appropriate example) clearly satisfies all requirements, but does never vanish (**1 point** for the correct proof that the example falsifies the claim).

d. *True* (**0.5 points**). Indeed, from the lecture we know that for f as given, if $f\big|_{\mathbf{C}} \subset \mathbf{C}$ then $f\big|_{\mathbf{C}}$ is a polynomial (**0.5 points**). Logically negating this statement yields if $f\big|_{\mathbf{C}}$ is not a polynomial, then $f(z) = \infty$ for some $z \in \mathbf{C}$ (**0.5 points**). Since such a z corresponds to an irremovable singularity of f (**0.5 points**), the claim follows.

e. *False* (**0.5 points**). Take any nonholomorphic function $f \colon \mathbf{C}^2 \to \mathbf{C}$ (e.g. with a jump) and $g \colon \mathbf{C}^2 \to \mathbf{C}^2$ to be a constant function (**0.5 points** for an appropriate example). Then $f \circ g$ is constant, hence holomorphic (**1 point** for the correct proof that the example falsifies the claim).

Problem 2. Infinite products $(3+4+3$ points$)$

The Eulerian Γ-function $\Gamma\colon \mathbf{C}\setminus\{0,-1,-2,\dots\}\to\mathbf{C}$ is defined by

$$\Gamma(z) := \frac{\mathrm{e}^{-\gamma z}}{z}\prod_{n=1}^{\infty}\Big[1+\frac{z}{n}\Big]^{-1}\mathrm{e}^{z/n}.$$

Here $\gamma\in\mathbf{R}$ is a normalization constant such that $\Gamma(1)=1$.

a. Show that the infinite product defining Γ is locally uniformly convergent on $\mathbf{C}\setminus\{-1,-2,\dots\}$. Moreover, show a constant γ with the stated properties exists. Deduce Γ is holomorphic on its domain of definition.

Hint. You can use the locally uniform convergence of the infinite product

$$\prod_{n=1}^{\infty}\Big[1+\frac{z}{n}\Big]\,\mathrm{e}^{-z/n}$$

on $\mathbf{C}$ without proof. You may want to consult Problem 1.

b. Show the identity

$$\gamma = \lim_{k\to\infty}\Big[\sum_{n=1}^{k}\frac{1}{n}-\log k\Big].$$

The limit on the right-hand side is called Euler–Mascheroni constant.

c. Show the recursion $\Gamma(z+1)=z\,\Gamma(z)$, where $z\in\mathbf{C}\setminus\{0,-1,-2,\dots\}$. Deduce Γ interpolates the factorial, viz. $\Gamma(m+1)=m!$ for every $m\in\mathbf{N}\cup\{0\}$.

Solutions. a. The infinite product from the hint vanishes precisely on $-\mathbf{N}$. Thus, it is nonconstant (**0.5 points**). Its partial products are clearly holomorphic (**0.5 points**). Problem 1.a. combines with the hint to yield the desired locally uniform convergence of the infinite product defining Γ.

To show the existence of γ, from the previous consideration the infinite product

$$\prod_{n=1}^{\infty}\Big[1+\frac{1}{n}\Big]^{-1}\mathrm{e}^{1/n}$$

converges. It is clearly a positive real number (**0.5 points**). Setting

$$\gamma := -\log\prod_{n=1}^{\infty}\Big[1+\frac{1}{n}\Big]\,\mathrm{e}^{-1/n},$$

it is straightforward to check γ obeys the desired normalization property (**0.5 points**).

We turn to holomorphy of Γ. The assignment $\mathrm{e}^{-\gamma z}/z$ depends holomorphically on $z\in\mathbf{C}\setminus\{0\}$ (**0.5 points**). The partial products from the product defining Γ are clearly holomorphic on $\mathbf{C}\setminus\{-1,-2,\dots\}$. By their locally uniform convergence, as shown in the lecture their infinite product is holomorphic on the same set, which is a domain (**0.5 points**). The product of these two terms is holomorphic on the intersection $\mathbf{C}\setminus\{0,-1,-2,\dots\}$ of their domains.

b. Thanks to the discussion from a., all limits appearing below will exist (**1 point** for a short justification of all limits, exchanges, etc.). Using a., the continuity of the logarithm, and logarithm rules (**1 point** for the first three equalities),

$$\begin{aligned}\gamma &= -\lim_{k\to\infty}\log\prod_{n=1}^{k}\Big[1+\frac{1}{n}\Big]\,\mathrm{e}^{-1/n}\\ &= -\lim_{k\to\infty}\sum_{n=1}^{k}\log\Big[\Big[\frac{n+1}{n}\Big]\,\mathrm{e}^{-1/n}\Big]\\ &= \lim_{k\to\infty}\sum_{n=1}^{k}\Big[\frac{1}{n}-\log(n+1)+\log n\Big]\\ &= \lim_{k\to\infty}\Big[\sum_{n=1}^{k}\frac{1}{n}-\log(k+1)\Big].\end{aligned}$$

The last identity comes from a telescopic sum (**1 point**). Since

$$\lim_{k\to\infty} \big[\log(k+1) - \log k\big] = \lim_{k\to\infty} \log\Big[1 + \frac{1}{k}\Big] = \log 1 = 0,$$

we can replace $\log(k+1)$ by $\log k$ in the above sum (**1 point**).

c. We start from the following identity implicitly shown in b.:

$$\gamma = \lim_{k\to\infty} \sum_{n=1}^{k} \Big[\frac{1}{n} - \log\Big[1 + \frac{1}{n}\Big]\Big].$$

Using this together with the definition of $\Gamma(z+1)$, we compute (**1 point**)

$$\begin{aligned}
\Gamma(z+1) &= \frac{\mathrm{e}^{-\gamma}\,\mathrm{e}^{-\gamma z}}{z+1} \prod_{n=1}^{\infty} \Big[1 + \frac{z+1}{n}\Big]^{-1} \mathrm{e}^{z/n}\,\mathrm{e}^{1/n} \\
&= \frac{\mathrm{e}^{-\gamma z}}{z+1} \prod_{n=1}^{\infty} \Big[1 + \frac{z+1}{n}\Big]^{-1} \Big[1 + \frac{1}{n}\Big] \mathrm{e}^{z/n} \\
&= \frac{\mathrm{e}^{-\gamma z}}{z+1} \prod_{n=1}^{\infty} \Big[1 + \frac{z}{n+1}\Big]^{-1} \mathrm{e}^{z/n} \\
&= z\,\Gamma(z).
\end{aligned}$$

To show the interpolation formula, we use induction on m. For $m = 0$ the claim $\Gamma(1) = 1 = 0!$ holds by the selected normalization (**1 point**). The previous recursion formula and the induction hypothesis combine to yield (**1 point**)

$$\Gamma(m+1) = m\,\Gamma(m) = m\,(m-1)! = m!.$$

Problem 3. Zeros and singularities ($3+3+4$ points)

Let $(a_n)_{n\in\mathbf{N}}$ constitute a sequence of mutually distinct elements of $\mathbf{C}$ without accumulation points.

a. By stating a general form of it and verifying all relevant hypotheses, given any $m \in \mathbf{N}$ show there exists an entire function $g_m\colon \mathbf{C} \to \mathbf{C}$ with the following properties: its set of zeros is precisely $\{a_n : n \in \mathbf{N}\}$ and for every $n \in \mathbf{N}$, the multiplicity of the zero a_n is m.

b. We fix three sequences $(\widetilde{p}_n)_{n\in\mathbf{N}}$, $(\widetilde{q}_n)_{n\in\mathbf{N}}$, and $(\widetilde{r}_n)_{n\in\mathbf{N}}$ in $\mathbf{C}$. By stating a general form of it and verifying all relevant hypotheses, show there exists a holomorphic function $h\colon \mathbf{C}\setminus\{a_n : n \in \mathbf{N}\} \to \mathbf{C}$ such that for every $n \in \mathbf{N}$, it has the following principal part around the point a_n:

$$\frac{\widetilde{r}_n}{z-a_n} + \frac{\widetilde{q}_n}{(z-a_n)^2} + \frac{\widetilde{p}_n}{(z-a_n)^3}.$$

c. We fix three sequences $(p_n)_{n\in\mathbf{N}}$, $(q_n)_{n\in\mathbf{N}}$, and $(r_n)_{n\in\mathbf{N}}$ in $\mathbf{C}$. Construct an entire function $f\colon \mathbf{C} \to \mathbf{C}$ with $f(a_n) = p_n$, $f'(a_n) = q_n$, and $f''(a_n) = r_n$ for every $n \in \mathbf{N}$. In particular, show your candidate has the desired properties.

Solutions. a. Assume first the sequence $(a_n)_{n\in\mathbf{N}}$ does not contain zero. Define a sequence $(b_n)_{n\in\mathbf{N}}$ as follows: the first m elements are equal to 0, the next m elements are equal to a_2, etc. (**0.5 points**). Then the sequence $(b_n)_{n\in\mathbf{N}}$ is clearly discrete (**0.5 points**). By construction, for every $n \in \mathbf{N}$ the number o_n of occurrences of a_n is exactly m; in particular, it is finite (**0.5 points**). The Weierstraß product theorem applies and yields the desired function. It can be taken of the form (**0.5 points**)

$$g_m(z) = \prod_{n=1}^{\infty} \Big[1 - \frac{z}{b_n}\Big] \exp\Big[\sum_{i=1}^{n} \frac{1}{i}\Big[\frac{z}{b_n}\Big]^i\Big].$$

If the sequence contains zero, one has to remove it from the above construction and eventually multiply the factor z^m to the above form of g_m (**1 point** for correct discussion of zero).

b. As in a. the sequence $(a_n)_{n\in\mathbf{N}}$ is discrete (**1 point**). Then by the Mittag-Leffler theorem (**1 point**) there exists a function $h\colon \mathbf{C}\setminus\{a_n : n\in\mathbf{N}\} \to \mathbf{C}$ with the given principal part Q_n at a_n for every $n \in \mathbf{N}$. It can be taken of the form

$$h(z) := \sum_{n=1}^{\infty} \big[Q_n - H_n\big],$$

where $H_n\colon \mathbf{C} \to \mathbf{C}$ is entire for every $n \in \mathbf{N}$ (**1 point**).

c. Let g_3 denote the entire function constructed in a. for m being 3. Since the multiplicity of a_n as its zero is equal to three for every $n \in \mathbf{N}$ by construction, the Laurent series of g_3 around a_n takes the form

$$g_3(z) = c_3\,(z-a_n)^3 + c_4\,(z-a_n)^4 + c_5\,(z-a_n)^5 + \sum_{k=4}^{\infty} c_k\,(z-a_n)^k,$$

where $c_3 \in \mathbf{C}\setminus\{0\}$ (**0.5 points**). Let us define three sequences $(\widetilde{p}_n)_{n\in\mathbf{N}}$, $(\widetilde{q}_n)_{n\in\mathbf{N}}$, $(\widetilde{r}_n)_{n\in\mathbf{N}}$ of numbers in $\mathbf{C}$ by (**1 point**)

$$\begin{aligned}
\widetilde{p}_n &:= p_n,\\
\widetilde{q}_n &:= q_n - \widetilde{p}_n - c_3 - c_4,\\
2\widetilde{r}_n &:= r_n - 2\big[\widetilde{p}_n + \widetilde{q}_n + c_3 + c_4 + c_5\big].
\end{aligned}$$

Let h denote the function constructed in b. with respect to this sequence. For every $n \in \mathbf{N}$, by construction its Laurent series expansion around a_n takes the form

$$\frac{\widetilde{r}_n}{z-a_n} + \frac{\widetilde{q}_n}{(z-a_n)^2} + \frac{\widetilde{p}_n}{(z-a_n)^3} + \sum_{k=0}^{\infty} d_k\,(z-a_n)^k.$$

We claim the product $f := g_3\,h$ (**0.5 points**), initially defined on $\mathbf{C}\setminus\{a_n : n \in \mathbf{N}\}$, has an entire extension which satisfies all desired properties. First, we observe f is meromorphic with singularities precisely at $\{a_n : n \in \mathbf{N}\}$ and the following Laurent series expansion around a_n for every $n \in \mathbf{N}$ (**0.5 points**):

$$\begin{aligned} f = \widetilde{p}_n + \big[c_4 + \widetilde{p}_n + c_3 + \widetilde{q}_n\big]\,(z - a_n) \\ + \big[c_5 + \widetilde{p}_n + c_4 + \widetilde{q}_n + c_3 + \widetilde{r}_n\big]\,(z - a_n)^2 + \sum_{k=3}^{\infty} e_k\,(z - a_n)^k. \end{aligned}$$

In particular, the singularity a_n is removable (**0.5 points**). By construction, the nonrelabeled entire extension of f satisfies all desired properties: it is holomorphic (**0.5 points**) and satisfies all pointwise requirements (**0.5 points**).

Problem 4. Picard's little theorem $(3 + 4 + 3$ points$)$

Given an open set $U \subset \mathbf{C}$, recall $\mathcal{H}(\overline{U})$ is the set of all complex functions $f\colon V \to \mathbf{C}$ which are holomorphic in a neighborhood $V \subset \mathbf{C}$ (possibly depending on f) of $\overline{U}$.

a. State Picard's little theorem.

b. Show there exists a function $R\colon \mathbf{C} \setminus \{0, 1\} \to (0, \infty)$ with the property that for every $a \in \mathbf{C} \setminus \{0, 1\}$,

$$\{f \in \mathcal{H}(\overline{B}_{R(a)}(0)) : f(0) = a,\ f'(0) = 1,\ f \text{ omits } 0 \text{ and } 1\} = \emptyset.$$

Hint. Set $R(a) := 3\,L(1/2, |a|)$, where L is given by Schottky's theorem.

c. Show the statement in b. implies Picard's little theorem.

Solutions. a. Let $f\colon \mathbf{C} \to \mathbf{C}$ be a nonconstant (**1 point**) entire (**1 point**) function. Then f assumes all except at most one values in $\mathbf{C}$ (**1 point**).

b. The function R from the hint is defined for every $a \in \mathbf{C} \setminus \{0\}$. Now let us fix $a \in \mathbf{C} \setminus \{0, 1\}$ and assume by contradiction that the class in question contains an element f. Since this function is holomorphic and satisfies $f(0) = a$ and $f'(0) = 1$, the Laurent series of f around zero takes the form (**1 point**)

$$f(z) = a + z + \sum_{k=2}^{\infty} a_k\, z^k.$$

Consider the function $g \in \mathcal{H}(\overline{B}_1(0))$ with $g(z) := f(R(a)\,z)$. It satisfies $g(0) = a$ and omits the values 0 and 1 (**1 point**). Schottky's theorem entails (**1 point**)

$$\sup |g|(\overline{B}_{1/2}(0)) \leq L\Big[\frac{1}{2}, |a|\Big] = \frac{R(a)}{3}.$$

On the other hand, by the Cauchy derivative formula we have (**1 point**)

$$R(a) = |g'(0)| = \Big|\frac{1}{2\pi\mathrm{i}} \int_{\partial B_{1/2}(0)} \frac{g(z)}{z^2}\,\mathrm{d}z\Big| \leq 2 \sup |g|(\overline{B}_{1/2}(0)) \leq \frac{2R(a)}{3}.$$

This contradicts the nonvanishing of R.

c. Let $f\colon \mathbf{C} \to \mathbf{C}$ be a nonconstant entire function. Assume to the contrary that f does not assume at least two distinct values. By an affine transformation, without loss of generality we may and will assume $f(\mathbf{C}) \cap \{0, 1\} = \emptyset$ (**1 point**). Choose $z_0 \in \mathbf{C}$ such that $f'(z_0) \neq 0$, which exists thanks to the nonconstancy of f, and set $a := f(z_0) \in \mathbf{C} \setminus \{0, 1\}$. Then the assignment $h(z) := f(z_0 + z/f'(z_0))$ is entire, satisfies $h(0) = a$ and $h'(0) = 1$, and omits the values 0 and 1 (**1 point**). This contradicts the emptiness of the class from b. (**1 point** for the correct use of a.).

Problem 5. Riemannian mapping theorem ($3 + 4 + 3$ points)

Let $G \subset \mathbf{C}$ be a simply connected domain which is invariant under complex conjugation, i.e. the inclusion $z \in G$ implies $\overline{z} \in G$. Let $a \in G \cap \mathbf{R}$ and let $f\colon G \to B_1(0)$ be a biholomorphic map such that $f(a) = 0$ and $f'(a) \in \mathbf{R}$ with $f'(a) > 0$. Lastly, define the function $g\colon G \to \mathbf{C}$ by

$$g(z) := \overline{f(\overline{z})}.$$

a. Show g is well-defined, holomorphic, and $g(G) = B_1(0)$.

   Hint. Verify the Cauchy–Riemann equations.

b. Show g coincides with f. Deduce the identity $f(G \cap \mathbf{R}) = B_1(0) \cap \mathbf{R}$.

c. Let $\mathbf{H}_+, \mathbf{H}_- \subset \mathbf{C}$ denote the upper and lower halfplane of $\mathbf{C}$ (without the real axis), respectively. Show either $f(G \cap \mathbf{H}_+) = B_1(0) \cap \mathbf{H}_+$ or $f(G \cap \mathbf{H}_+) = B_1(0) \cap \mathbf{H}_-$ holds.

Solutions. a. Since the function f is defined on G yet G is invariant under complex conjugation, g is well-defined (**0.5 points**).

Next, we claim holomorphy of g. We regard f, $\Re f$, and $\Im f$ as maps from $\mathbf{R}^2$ to $\mathbf{R}^2$ in the usual way through $f(x, y) := \Re f(x, y) + \mathrm{i}\,\Im f(x, y)$. Doing the same for g, it follows from the definition of g that

$$g(x, y) = \Re f(x, -y) - \mathrm{i}\,\Im f(x, -y).$$

In particular, g is C^1 as a map from $\mathbf{R}^2$ to $\mathbf{R}^2$ (**0.5 points**). By the chain rule, we compute the following identities for every appropriate $x, y \in \mathbf{R}^2$ (**0.5 points**):

$$\begin{aligned}
\partial_1 \Re g(x, y) &= \partial_1 \Re f(x, -y),\\
\partial_2 \Im g(x, y) &= \partial_2 \Re f(x, -y),\\
\partial_2 \Re g(x, y) &= -\partial_2 \Re f(x, -y),\\
\partial_1 \Im g(x, y) &= -\partial_1 \Im f(x, -y).
\end{aligned}$$

In particular, since f obeys the Cauchy–Riemann equations, so does g (**0.5 points**). These observations imply holomorphy of g.

Lastly, we show $g(G) = B_1(0)$. The inclusion "$\subset$" follows since f maps G to $B_1(0)$ and the latter is invariant under complex conjugation (**0.5 points**). For the inclusion "$\supset$", let $w \in B_1(0)$. As $\overline{w} \in B_1(0)$ and f is bijective, there exists a point $x \in G$ with $f(x) = \overline{w}$. In turn, by assumption on G the point $\overline{x}$ belongs to G. By construction, we thus infer $g(\overline{x}) = \overline{f(x)} = w$ (**0.5 points**).

b. By a., we already know g is holomorphic. Furthermore, it clearly inherits the injectivity of f (**0.5 points**). Since $g(G) = B_1(0)$ by a. again, from the lecture we know that g is biholomorphic (**0.5 points**). Moreover, we clearly have $g(a) = \overline{f(a)} = 0$ (**0.5 points**). With the identification from a., we compute (**1 point**)

$$\begin{aligned}
g'(a) &= \lim_{\substack{h \to 0,\\ h \in \mathbf{R}\setminus\{0\}}} \frac{g(a + h) - g(a)}{h}\\
&= \lim_{\substack{h \to 0,\\ h \in \mathbf{R}\setminus\{0\}}} \frac{\Re g(\Re a + h, \Im a) - \Re g(\Re a, \Im a)}{h}\\
&\qquad + \mathrm{i} \lim_{\substack{h \to 0,\\ h \in \mathbf{R}\setminus\{0\}}} \frac{\Im g(\Re a + h, \Im a) - \Im g(\Re a, \Im a)}{h}\\
&= \partial_1 \Re g(\Re a, 0) + \mathrm{i}\, \partial_1 \Im g(\Re a, 0)\\
&= \partial_1 \Re f(\Re a, 0) - \mathrm{i}\, \partial_1 \Im f(\Re a, 0)\\
&= \partial_1 \Re f(\Re a, 0).
\end{aligned}$$

Here, we have used $\partial_1 \Im f(\Re a, 0) = 0$, as implied by an analogous computation for the real number $f'(a)$. This readily yields $g'(a) \in \mathbf{R}$ and $g'(a) > 0$ (**0.5 points**). By the uniqueness statement from the Riemannian mapping theorem, this entails the identity of the functions g and f.

We turn to the identity $f(G \cap \mathbf{R}) = B_1(0) \cap \mathbf{R}$. From the previous paragraph, we obtain $f(\overline{z}) = \overline{f(z)}$ for every $z \in G$. In particular, if $z \in G \cap \mathbf{R}$ we get $f(z) = \overline{f(z)}$, which entails $f(z) \in B_1(0) \cap \mathbf{R}$ and thus the inclusion "$\subset$" (**0.5 points**). For the inclusion, recall from item a. that for every $w \in B_1(0) \cap \mathbf{R}$ there exists $y \in G$ such that $f(y) = w$. Complex conjugation of this identity yields $w = \overline{f(y)} = f(\overline{y})$ and therefore $f(y) = f(\overline{y})$. Bijectivity of f entails $y = \bar{y}$ and thus $y \in G \cap \mathbf{R}$, and in turn $w \in f(G \cap \mathbf{R})$ (**0.5 points**).

c. We first show $G \cap \mathbf{H}_+$ is path-connected. Indeed, since G is a domain any two points therein can be connected by a continuous curve with values in G. Since G is invariant under complex conjugation, flipping the imaginary part of such a curve every time it takes values in $G \cap \mathbf{H}_-$ yields the connecting curve can be chosen to take values in $G \cap \overline{\mathbf{H}}_+$ (**0.5 points**). By a covering argument (using compactness of the curve segment and openness of G), each time this curve meets $G \cap \mathbf{R}$ it can be "pushed up" slightly to yield a resulting connecting curve passing through $G \cap \mathbf{H}_+$ (**0.5 points**).

We first claim $f(G \cap \mathbf{H}_+) \subset B_1(0) \cap \mathbf{H}_+$ or $f(G \cap \mathbf{H}_+) \subset B_1(0) \cap \mathbf{H}_-$. Assume to the contrary that this is not the case. By b. and bijectivity of f, this means there are two necessarily distinct points $y_+, y_- \in G \cap \mathbf{H}_+$ such that $f(y_+) \in B_1(0) \cap \mathbf{H}_+$ and $f(y_-) \in B_1(0) \cap \mathbf{H}_-$. Since $G \cap \mathbf{H}_+$ is path-connected, there exists a continuous curve $\gamma\colon [0,1] \to G \cap \mathbf{H}_+$ from y_+ to y_-. By the mean value theorem, there exists $t \in (0,1)$ such that $f \circ \gamma(t) \in B_1(0) \cap \mathbf{R}$ (**0.5 points**). At such a point, by b. it follows $\gamma(t) \in G \cap \mathbf{R}$. This contradicts the fact that γ never leaves $\mathbf{H}_+$ (**0.5 points**).

An analogous argument as above yields the inclusion $f(G \cap \mathbf{H}_-) \subset B_1(0) \cap \mathbf{H}_+$ or $f(G \cap \mathbf{H}_-) \subset B_1(0) \cap \mathbf{H}_-$. A totality argument forces equalities (**1 point**).

Problem 6. Möbius transformations $(3+4+3$ points$)$

Let $f\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be a Möbius transformation. That is, there are constants $a, b, c, d \in \mathbf{C}$ with $ad - bc \neq 0$ such that for every $z \in \widehat{\mathbf{C}}$,

$$f(z) = \begin{cases} \dfrac{a}{c} & \text{if } z = \infty, \\ \infty & \text{if } z = -\dfrac{d}{c}, \\ \dfrac{az+b}{cz+d} & \text{otherwise.} \end{cases}$$

a. Show f is the identity if and only if $a = d$ and $b = c = 0$.

   Hint. Insert appropriate arguments $z \in \widehat{\mathbf{C}}$.

b. Show f has exactly one or exactly two fixed points unless it is the identity. Give an example of a Möbius transformation with exactly one and exactly two fixed points, respectively.

c. Show for every $p, q \in \widehat{\mathbf{C}}$ there exists a Möbius transformation $f\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ such that $f(p) = q$.

   Hint. First consider a specific choice for p. Why is this sufficient?

Solutions. a. It is evident that if $a = d$ and $b = c = 0$, which also rules out the vanishing of a and d by the constraint $ad - bc \neq 0$ (**0.5 points**), then f is the identity map (**0.5 point**).

Conversely, suppose $f(z) = z$ for every $z \in \widehat{\mathbf{C}}$. First, the constraint $ad - bc \neq 0$ (**0.5 points** for the careful check of the constraint throughout) rules out the case $a = c = 0$, hence $a \neq 0$ or $c \neq 0$. The identity $f(\infty) = \infty$ then forces $c = 0$ (**0.5 points**). Second, the constraint $ad - bc \neq 0$ rules out the case $b = d = 0$, hence $b \neq 0$ or $d \neq 0$. The identity $f(0) = 0$ then forces $b = 0$ (**0.5 points**). Third, in turn the constraint translates into $ad \neq 0$, which forces $a \neq 0$ and $d \neq 0$. The identity $f(1) = 1$ then forces $1 = a/d$, or equivalently $a = d$ (**0.5 points**).

b. Assume f is not the identity map. Suppose first ∞ is a fixed point of f. As in a., this forces $c = 0$ (**0.5 points**). The constraint $ad - bc \neq 0$ rules out the vanishing of d. This means that every fixed point $z \in \mathbf{C}$ obeys $dz = az + b$ after an elementary algebraic manipulation, or equivalently $(a - d)z = -b$. If $a = d$, this equation would force $b = 0$ and hence f was the identity map by a., which has been excluded (**0.5 points**). Hence, necessarily $a \neq d$. This discussion yields exactly two fixed points ∞ and $-b/(a - d)$ of f (**0.5 points**).

Now we suppose ∞ is not a fixed point of f. This forces $c \neq 0$ (**0.5 points**). By our hypothesis, the fixed points z of f can only lie in $\mathbf{C}$ and necessarily satisfy the following identity (**0.5 points**):

$$\frac{az+b}{cz+d} = z.$$

Since $z \neq \infty$, this forces $cz + d \neq 0$. The above equation becomes

$$az + b = (cz + b)z.$$

This forms a quadratic equation with leading coefficient $c \neq 0$. By the fundamental theorem of algebra, it has either one or two complex solutions z (**0.5 points**).

An example with exactly one fixed point, namely ∞, is the translation $z + 1$ (**0.5 points** for a valid example). An example with exactly two fixed points, namely 0 and ∞, is the scaling $2z$ (**0.5 points** for a valid example). In both cases, the determinant constraint $ad - bc \neq 0$ holds.

c. Given $q \in \widehat{\mathbf{C}}$, we first construct a Möbius transformation $f\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ such that $f(0) = q$. If $q = 0$ we can simply take the identity map (**0.5 points**). If $q \in \mathbf{C} \setminus \{0\}$ we can e.g. take $a = d = 1$, $b = q$, and $c = 2/q$ above (**0.5 points** for an appropriate choice of the coefficients). If $q = \infty$ we can take $a = b = c = 1$ and $d = 0$ above (**0.5 points** for an appropriate choice of the coefficients). In all these cases, the determinant constraint $ad - bc \neq 0$ holds.

Now let $p, q \in \widehat{\mathbf{C}}$ be arbitrary. By the previous paragraph, there exists a Möbius transformation $f\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ with $f(0) = q$ (**0.5 points**). Moreover, there exists a Möbius transformation $h\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ with $h(0) = p$. As the class of Möbius transformations forms a group with respect to composition, the inverse g of h is a Möbius transformation with $g(p) = 0$ (**0.5 points**). Again using the previous group structure, the composition $f \circ g$ is thus a Möbius transformation with $f \circ g(p) = f(0) = q$ (**0.5 points**).