MATH 327 - TOPICS IN COMPLEX ANALYSIS
PRACTICE EXAM
TIME: 2 HOURS

Problem 1. True or false $(2+2+2+2+2$ points)

For each of the following statements, decide whether it is true or false. Justify each statement you deem true with a short proof. For each claim you deem false, give an explicit counterexample. Please read the claims carefully to not miss a word!

a. Let $D \subset \mathbf{C}$ be a domain. Let $(f_n)_{n\in\mathbf{N}}$ be a sequence of nowhere vanishing holomorphic functions $f_n : D \to \mathbf{C}$ converging locally uniformly to a nonconstant function $f : D \to \mathbf{C}$. Then $(1/f_n)_{n\in\mathbf{N}}$ converges locally uniformly to $1/f$ on D.

b. Let $S \subset \mathbf{C}$ be a countable set. Then there is an entire function $f : \mathbf{C} \to \mathbf{C}$ whose set of zeros is exactly S.

c. Let $f : \mathbf{C} \to \mathbf{C}$ be a nonconstant entire function with $f(0) = 1$ and $f'(0) = 0$. Then f has a zero.

d. Let $f : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be a nonconstant holomorphic function. Assume $f\big|_{\mathbf{C}}$ is not a polynomial. Then f has a pole or an essential singularity.

e. Assume the function $f : \mathbf{C}^n \to \mathbf{C}$ is not holomorphic while $g : \mathbf{C}^n \to \mathbf{C}^n$ is, where n is at least 2. Then the composition $f \circ g$ is not holomorphic.

Problem 2. Infinite products $(3+4+3$ points)

The Eulerian Γ-function $\Gamma : \mathbf{C} \setminus \{0, -1, -2, \dots\} \to \mathbf{C}$ is defined by

$$\Gamma(z) := \frac{\mathrm{e}^{-\gamma z}}{z} \prod_{n=1}^{\infty} \Big[1 + \frac{z}{n}\Big]^{-1} \mathrm{e}^{z/n}.$$

Here $\gamma \in \mathbf{R}$ is a normalization constant such that $\Gamma(1) = 1$.

a. Show that the infinite product defining Γ is locally uniformly convergent on $\mathbf{C} \setminus \{-1, -2, \dots\}$. Moreover, show a constant γ with the stated properties exists. Deduce Γ is holomorphic on its domain of definition.

   Hint. You can use the locally uniform convergence of the infinite product

$$\prod_{n=1}^{\infty} \Big[1 + \frac{z}{n}\Big] \mathrm{e}^{-z/n}$$

   on $\mathbf{C}$ without proof. You may want to consult Problem 1.

b. Show the identity

$$\gamma = \lim_{k\to\infty} \Big[\sum_{n=1}^{k} \frac{1}{n} - \log k\Big].$$

   The limit on the right-hand side is called Euler–Mascheroni constant.

c. Show the recursion $\Gamma(z+1) = z\,\Gamma(z)$, where $z \in \mathbf{C} \setminus \{0, -1, -2, \dots\}$. Deduce Γ interpolates the factorial, viz. $\Gamma(m+1) = m!$ for every $m \in \mathbf{N} \cup \{0\}$.

Problem 3. Zeros and singularities $(3+3+4$ points)

Let $(a_n)_{n\in\mathbf{N}}$ constitute a sequence of mutually distinct elements of $\mathbf{C}$ without accumulation points.

a. By stating a general form of it and verifying all relevant hypotheses, given any $m \in \mathbf{N}$ show there exists an entire function $g_m : \mathbf{C} \to \mathbf{C}$ with the following properties: its set of zeros is precisely $\{a_n : n \in \mathbf{N}\}$ and for every $n \in \mathbf{N}$, the multiplicity of the zero a_n is m.

b. We fix three sequences $(\widetilde{p}_n)_{n\in\mathbf{N}}$, $(\widetilde{q}_n)_{n\in\mathbf{N}}$, and $(\widetilde{r}_n)_{n\in\mathbf{N}}$ in $\mathbf{C}$. By stating a general form of it and verifying all relevant hypotheses, show there exists a holomorphic function $h : \mathbf{C} \setminus \{a_n : n \in \mathbf{N}\} \to \mathbf{C}$ such that for every $n \in \mathbf{N}$, it has the following principal part around the point a_n:

$$\frac{\widetilde{r}_n}{z - a_n} + \frac{\widetilde{q}_n}{(z-a_n)^2} + \frac{\widetilde{p}_n}{(z-a_n)^3}.$$

c. We fix three sequences $(p_n)_{n\in\mathbf{N}}$, $(q_n)_{n\in\mathbf{N}}$, and $(r_n)_{n\in\mathbf{N}}$ in $\mathbf{C}$. Construct an entire function $f\colon \mathbf{C}\to\mathbf{C}$ with $f(a_n) = p_n$, $f'(a_n) = q_n$, and $f''(a_n) = r_n$ for every $n \in \mathbf{N}$. In particular, show your candidate has the desired properties.

Problem 4. Picard's little theorem $(3 + 4 + 3$ points)

Given an open set $U \subset \mathbf{C}$, recall $\mathcal{H}(\overline{U})$ is the set of all complex functions $f\colon V \to \mathbf{C}$ which are holomorphic in a neighborhood $V \subset \mathbf{C}$ (possibly depending on f) of $\overline{U}$.

a. State Picard's little theorem.

b. Show there exists a function $R\colon \mathbf{C}\setminus\{0,1\} \to (0,\infty)$ with the property that for every $a \in \mathbf{C}\setminus\{0,1\}$,

$$\{f \in \mathcal{H}(\overline{B}_{R(a)}(0)) : f(0) = a,\ f'(0) = 1,\ f \text{ omits } 0 \text{ and } 1\} = \emptyset.$$

Hint. Set $R(a) := 3\,L(1/2, |a|)$, where L is given by Schottky's theorem.

c. Show the statement in b. implies Picard's little theorem.

Problem 5. Riemannian mapping theorem $(3 + 4 + 3$ points)

Let $G \subset \mathbf{C}$ be a simply connected domain which is invariant under complex conjugation, i.e. the inclusion $z \in G$ implies $\overline{z} \in G$. Let $a \in G \cap \mathbf{R}$ and let $f\colon G \to B_1(0)$ be a biholomorphic map such that $f(a) = 0$ and $f'(a) \in \mathbf{R}$ with $f'(a) > 0$. Lastly, define the function $g\colon G \to \mathbf{C}$ by

$$g(z) := \overline{f(\overline{z})}.$$

a. Show g is well-defined, holomorphic, and $g(G) = B_1(0)$.

Hint. Verify the Cauchy–Riemann equations.

b. Show g coincides with f. Deduce the identity $f(G \cap \mathbf{R}) = B_1(0) \cap \mathbf{R}$.

c. Let $\mathbf{H}_+, \mathbf{H}_- \subset \mathbf{C}$ denote the upper and lower halfplane of $\mathbf{C}$ (without the real axis), respectively. Show either $f(G \cap \mathbf{H}_+) = B_1(0) \cap \mathbf{H}_+$ or $f(G \cap \mathbf{H}_+) = B_1(0) \cap \mathbf{H}_-$ holds.

Problem 6. Möbius transformations $(3 + 4 + 3$ points)

Let $f\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be a Möbius transformation. That is, there are constants $a, b, c, d \in \mathbf{C}$ with $ad - bc \neq 0$ such that for every $z \in \widehat{\mathbf{C}}$,

$$f(z) = \begin{cases} \dfrac{a}{c} & \text{if } z = \infty, \\ \infty & \text{if } z = -\dfrac{d}{c}, \\ \dfrac{az+b}{cz+d} & \text{otherwise.} \end{cases}$$

a. Show f is the identity if and only if $a = d$ and $b = c = 0$.

Hint. Insert appropriate arguments $z \in \widehat{\mathbf{C}}$.

b. Show f has exactly one or exactly two fixed points unless it is the identity. Give an example of a Möbius transformation with exactly one and exactly two fixed points, respectively.

c. Show for every $p, q \in \widehat{\mathbf{C}}$ there exists a Möbius transformation $f\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ such that $f(p) = q$.

Hint. First consider a specific choice for p. Why is this sufficient?